Week 9: 9.5: 9, 15, 23, 24

9.6: 11, 19, 25, 36

9. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n}$ is an alternating series with $a_n = \frac{n}{\ln n}$, but $\lim_{n\to\infty} \frac{(-1)^n n}{\ln n}$ does not exist because, using l'Hôpital's Rule, $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$. Thus, the series diverges by the Divergence Test.

15. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin \frac{(2n-1)\pi}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is convergent (see Exercise 5).

23. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ is an alternating series with $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$. Since $\frac{a_{n+1}}{a_n} = \frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n+1}+\sqrt{n+2}} < 1$ for $n \ge 1$, we see that $a_{n+1} < a_n$ for $n \ge 1$, and so $\{a_n\}$ is decreasing for $n \ge 1$. Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = 0$, so the AST implies that the given series converges.

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[n]{n}}$ is an alternating series with $a_n = \frac{1}{\sqrt[n]{n}}$. Consider $y = f(x) = \sqrt[x]{x} = x^{1/x}$, so $\ln y = \frac{\ln x}{x}$. Then, using l'Hôpital's Rule, $\ln\left(\lim_{x\to\infty} y\right) = \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow \lim_{x\to\infty} \sqrt[x]{x} = 1$, showing that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$. Therefore, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{\sqrt[n]{n}}$ does not exist, and the given series diverges by the Divergence Test.

11. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$. Using the Ratio Test with $a_n = \frac{n!}{e^n}$, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right] = \lim_{n\to\infty} \frac{n+1}{e} = \infty$, so the series diverges.

19. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{2^n}$. Consider $\sum_{n=2}^{\infty} \left|\frac{(-1)^n \ln n}{2^n}\right| = \sum_{n=2}^{\infty} \frac{\ln n}{2^n}$. Since $\frac{\ln n}{2^n} < \frac{n}{2^n}$ and $\sum_{n=2}^{\infty} \frac{n}{2^n}$ converges (see Exercise 9.3.28), the Comparison Test implies that the given series converges absolutely.

25. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$. We use the Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left[\frac{1}{(\ln n)^n}\right]^{1/n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$, so the series converges absolutely.

36. Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. We find that for $p \neq 0$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$, so the Ratio Test is inconclusive. The case $p = 0$ is trivial: $\sum 1$ evidently diverges.