Week 7: 9.3: 4, 6, 16, 23

9.4: 9, 11, 17, 19

4. $\sum_{n=1}^{\infty} ne^{-n}$. Let $f(x) = xe^{-x}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.
$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx = \lim_{b\to\infty} \left[-(x+1)e^{-x}\right]_1^b = \lim_{b\to\infty} \left[-(b+1)e^{-b} + 2e^{-1}\right] = 2/e$ (using l'Hôpital's Rule). Since $\int_1^\infty xe^{-x}\,dx$ converges, so does the series.

6. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{3+4n}$. Let $f(x) = \frac{1}{3+4x}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$. $I = \int_1^\infty \frac{dx}{3+4x} = \lim_{b\to\infty} \int_1^b \frac{dx}{3+4x} = \lim_{b\to\infty} \frac{1}{4}\left[\ln(3+4b) - \ln 7\right] = \infty$. Since I is divergent, so is the series.

16. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^2+1}}$. Let $f(x) = \frac{x}{\sqrt{2x^2+1}}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.
$I = \int_1^\infty \frac{x\,dx}{\sqrt{2x^2+1}} = \lim_{b\to\infty} \int_1^b x\left(2x^2+1\right)^{-1/2} dx = \lim_{b\to\infty} \left(\frac{1}{2}\sqrt{2b^2+1} - \frac{1}{2}\sqrt{3}\right) = \infty$. Since I diverges, so does the series.

23. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Let $f(x) = \frac{1}{x(\ln x)^2}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$.
$\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \left[-\frac{1}{\ln b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$, so the series converges.

9. $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$.

11. $\frac{2+\sin n}{3^n} < \frac{3}{3^n} = \frac{1}{3^{n-1}}$ for $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{3}{3^n}$ is convergent, so is $\sum_{n=1}^{\infty} \frac{2+\sin n}{3^n}$.

17. If n is large, then $a_n = \frac{3n^2+1}{2n^5+n+2}$ behaves like $\frac{3n^2}{2n^5} = \frac{3}{2n^3}$, so take $b_n = \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n^2+1}{2n^5+n+2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{3n^5+n^3}{2n^5+n+2} = \frac{3}{2} > 0, \text{ so } \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{3n^2+1}{2n^5+n+2} \text{ converges.}$$

19. If n is large, then $a_n = \frac{1}{\sqrt{n^3-n-1}}$ behaves like $\frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} = b_n$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3-n-1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3-n-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}-\frac{1}{n^3}}} = 1 > 0, \text{ so } \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} \text{ converges} \Rightarrow$$
$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-n-1}} \text{ converges.}$$