Week 10: 9.7: 11, 15, 17, 21

9.8: 3, 5, 16, 29

10.2: 本節沒有作業

11. Let $u_n = \frac{e^n x^n}{n}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{e^n x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)e\,|x| = e\,|x|$, so the radius of convergence is $\frac{1}{e}$ and the series converges for $-\frac{1}{e} < x < \frac{1}{e}$. At $x = -\frac{1}{e}$, the series is $\sum\frac{(-1)^n}{n}$, which converges, and at $x = \frac{1}{e}$ it is $\sum\frac{1}{n}$, which diverges. Thus, the interval of convergence is $\left[-\frac{1}{e}, \frac{1}{e}\right)$.

15. Let $u_n = \frac{(-1)^{n-1}(x-2)^n}{n\cdot 3^n}$. Then

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(x-2)^{n+1}}{(n+1)\,3^{n+1}}\cdot\frac{n\cdot 3^n}{(-1)^{n-1}(x-2)^n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)|x-2| = \frac{1}{3}|x-2| < 1$
$\Leftrightarrow |x-2| < 3$, so the radius of convergence is 3 and the series converges on $(-1,5)$. At $x=-1$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}$, which diverges, and at $x=5$ it is $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$, which converges. Thus, the interval of convergence is $(-1,5]$.

17. Let $u_n = \frac{(-1)^n n(x-1)^n}{n^2+1}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x-1)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(-1)^n n(x-1)^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{n^2+1}{n^2+2n+2}\right)|x-1|$$
$$= |x-1|$$

so $R=1$ and the series converges on $(0,2)$. At $x=0$ the series is $\sum_{n=0}^{\infty}\frac{(-1)^n n(-1)^n}{n^2+1} = \sum_{n=0}^{\infty}\frac{n}{n^2+1}$, which diverges, and at $x=2$ it is $\sum_{n=0}^{\infty}\frac{(-1)^n n}{n^2+1}$, which converges. Thus, the interval of convergence is $(0,2]$.

21. Let $u_n = \frac{2^n(x+2)^n}{n^n}$. Then

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x+2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n(x+2)^n}\right| = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)\left(\frac{n}{n+1}\right)^n|x+2| = 0$ for any real x since $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = \frac{1}{e}$. Thus, $R=\infty$ and the interval of convergence is $(-\infty,\infty)$.

3. $f(x) = e^x,\ f'(x) = e^x,\ f''(x) = e^x,\ \ldots,\ f^{(n)}(x) = e^x,\ \ldots$

$f(2) = e^2,\ f'(2) = e^2,\ f''(2) = e^2,\ \ldots,\ f^{(n)}(2) = e^2,\ \ldots$

The required Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n = e^2\left[1 + (x-2) + \frac{1}{2!}(x-2)^2 + \cdots + \frac{1}{n!}(x-2)^n + \cdots\right].$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{e^2(x-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^2(x-2)^n}\right| = \lim_{n\to\infty}\left(\frac{1}{n+1}\right)|x-2| = 0, \text{ so } R = \infty.$$

5. $f(x) = \sin 2x,\ f'(x) = 2\cos 2x,\ f''(x) = (-1)^1 2^2 \sin 2x,\ f'''(x) = (-1)^1 2^3 \cos 2x,\ f^{(4)}(x) = (-1)^2 2^4 \sin 2x, \ldots$

$f(0) = 0, \quad f'(0) = 2, \quad f''(0) = 0, \quad f'''(0) = -2^3, \quad f^{(4)}(0) = 0, \quad \ldots$

The required Maclaurin series is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 2x - \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 - \cdots + \frac{(-1)^k 2^{2k+1}}{(2k+1)!}x^{2k+1} + \cdots.$

Note that we have changed the index variable to k, where $n = 2k + 1$.

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{2k+3}x^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{(-1)^k(2)^{2k+1}x^{2k+1}}\right| = \lim_{k\to\infty}\frac{4}{(2k+2)(2k+3)}|x|^2 = 0, \text{ so } R = \infty.$$

16. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\cdot\dfrac{1}{1+\left(\frac{x}{2}\right)^2} = \dfrac{1}{4}\cdot\dfrac{1}{1-\left[-\left(\frac{x}{2}\right)^2\right]} = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\left[-\left(\frac{x}{2}\right)^2\right]^n = \dfrac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n}$. The series converges if $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$, so $R = 2$.

29. $f(x) = \dfrac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + \dfrac{(-2)(-3)}{2!}x^2 + \cdots + \dfrac{(-2)(-2-1)\cdots(-2-n+1)}{n!}x^n + \cdots$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot(n+1)x^n}{n!} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \text{ with } R = 1.$$