Homework-9

Exercises of Section(11.5 , 12.1-12.2)

2018.05.22

- **(Section 11.5)**

(11). Find parametric and symmetric equations of the line passing through the point $(1, 2, -1)$ and parallel to the line with parametric equations $x = -1 + t$, $y = 2 + 2t$ and $z = -2 - 3t$. At what points does the line intersect the coordinates planes ?

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◦ Sol :

The direction of the given line is the same as the vector $\vec{v} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$, so the parametric equation of line are

$$x = 1 + t \ , \ y = 2 + 2t \ , \ z = -1 - 3t$$

and the symmetric equations are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+1}{-3}$$

Let $x = 0$ then $t = -1 \ \Rightarrow \ y = 0 \quad ; \quad z = 2$

So the line intersects the yz-plane at point$(0, 0, 2)$.

Let $y = 0$ then $t = -1 \ \Rightarrow \ x = 0 \quad ; \quad z = 2$

So the line intersects the xz-plane at point$(0, 0, 2)$.

Let $z = 0$ then $t = -1/3 \ \Rightarrow \ x = 2/3 \quad ; \quad z = 4/3$

So the line intersects the xz-plane at point$(2/3, 4/3, 0)$.

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(19). Determine whether the line L_1 and L_2 intersect. If they do intersect, find the angle between them.

$$L_1 : x = 1 - t \ , \ y = 3 - 2t \ , \ z = t$$

$$L_2 : x = 2 + 3t \ , \ y = 3 + 2t \ , \ z = 1 + t$$

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∘ Sol :

Suppose the two lines intersect at a point then there exists t_1 and t_2 so that

$$\begin{cases} 1 - t_1 = 2 + 3t_2 \\ 3 - 2t_1 = 3 + 2t_2 \\ t_1 = 1 + t_2 \end{cases} \Rightarrow \begin{cases} t_1 + 3t_2 = -1 \\ 2t_1 + 2t_2 = 0 \\ t_1 - t_2 = 1 \end{cases} \Rightarrow t_1 = 1/2 \ , \ t_2 = -1/2$$

Hence the two lines intersect at $(1/2, 2, 1/2)$.

A vector parallel to L_1 is $\vec{v}_1 = -\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and a vector parallel to L_2 is $\vec{v}_2 = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

The angle between them is

$$\theta = \cos^{-1}\left(\frac{|\vec{v}_1 \cdot \vec{v}_2|}{|\vec{v}_1||\vec{v}_2|}\right) = \cos^{-1}\left(\frac{6}{\sqrt{6}\sqrt{14}}\right) = \cos^{-1}\left(\sqrt{\frac{3}{7}}\right) \cong 49.1^\circ$$

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(43). Find the angle between the plane and the line

$$\boldsymbol{x + y + 2z = 6 \quad ; \quad x = 1 + t \ , \ y = 2 + t \ , \ z = -1 + t}$$

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∘ Sol :

The normal vector of the plane is $\vec{n} = \langle 1, 1, 2\rangle$ and a vector parallel to the line is $\vec{v} = \langle 1, 1, 1\rangle$, so the angle between the line and the normal vector is

$$\theta = \cos^{-1}\left(\frac{|\vec{n} \cdot \vec{v}|}{|\vec{n}||\vec{v}|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{6}\sqrt{3}}\right) \cong 19.5^\circ$$

Hence the angle between the plane and the line is about

$$90^\circ - 19.5^\circ = 70.5^\circ$$

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(49). Find an equation of the plane that contains the lines given by

$$\boldsymbol{L_1 \ : \ x = -1 + 2t \ , \ y = 2 - 3t \ , \ z = 1 + t}$$

$$\boldsymbol{L_2 \ : \ x = 2 - t \ , \ y = 1 - 2t \ , \ z = 5 - 3t}$$

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∘ Sol :

A vector parallel to L_1 is $\vec{v}_1 = \langle 2, -3, 1\rangle$ and a vector parallel to L_2 is $\vec{v}_2 = \langle -1, -2, -3\rangle$. Hence the normal vector of the plane is

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ -1 & -2 & -3 \end{vmatrix} = 11\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$$

Packing a point in the plane by taking $t = 0$, obtaining (-1,2,1) then the equation of the plane is

$$11(x+1) + 5(y-2) - 7(z-1) = 0 \ \Rightarrow \ 11x + 5y - 7z = -8$$

• (Section 12.1)

(38) Find the limit

$$\lim_{t\to 0} \left\langle e^{-t} \ , \ \frac{\sin t}{t} \ , \ \cos t \right\rangle = ?$$

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∘ Sol :

$$\lim_{t\to 0} \left\langle e^{-t}, \frac{\sin t}{t}, \cos t \right\rangle = \left\langle \lim_{t\to 0} e^{-t} \ , \ \lim_{t\to 0} \frac{\sin t}{t} \ , \ \lim_{t\to 0} \cos t \right\rangle = \left\langle 1, 1, 1 \right\rangle$$

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(42) Find the limit

$$\lim_{t\to -\infty} \left[\left(\frac{t+1}{2t+1} \right) \hat{\imath} + e^{2t}\hat{\jmath} + \tan^{-1} t\hat{k} \right]$$

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∘ Sol :

$$\lim_{t\to -\infty} \left[\left(\frac{t+1}{2t+1} \right) \hat{\imath} + e^{2t}\hat{\jmath} + \tan^{-1} t\hat{k} \right] = \lim_{t\to -\infty} \left(\frac{t+1}{2t+1} \right) \hat{\imath} + \lim_{t\to -\infty} e^{2t}\hat{\jmath} + \lim_{t\to -\infty} \tan^{-1} t\hat{k} = \frac{1}{2}\hat{\imath} - \frac{\pi}{2}\hat{k}$$

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(56) Evaluate

$$\lim_{t\to 0}\left\langle \frac{(t+h)^2-t^2}{h}\ ,\ \frac{\cos(t+h)-\cos t}{h}\ ,\ \frac{e^{t+h}-e^t}{h}\right\rangle = ?$$

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○ Sol :

$$\lim_{t\to 0}\left\langle \frac{(t+h)^2-t^2}{h}\ ,\ \frac{\cos(t+h)-\cos t}{h}\ ,\ \frac{e^{t+h}-e^t}{h}\right\rangle$$

$$= \left\langle \lim_{t\to 0}\frac{(t+h)^2-t^2}{h}\ ,\ \lim_{t\to 0}\frac{\cos(t+h)-\cos t}{h}\ ,\ \lim_{t\to 0}\frac{e^{t+h}-e^t}{h}\right\rangle$$

By using the definition of derivative.

$$= \left\langle 2t\ ,\ -\sin t\ ,\ e^t \right\rangle$$

- **(Section 12.2)**

(14). Find $\vec{r}(t)$ and $\vec{r}'(t)$ at the given value of t. Sketch the curve define by $\vec{r}$ and the vectors $\vec{r}(t)$ and $\vec{r}'(t)$ on the same set of axes.

$$\vec{r}(t) = \langle e^t\ ,\ e^{-2t}\rangle \quad ; \quad t = 0$$

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○ Sol :

(a). $\vec{r}(t) = \langle e^t\ ,\ e^{-2t}\rangle \Rightarrow \vec{r}(0) = \langle 1\ ,\ 1\rangle \quad ; \quad \vec{r}'(t) = \langle e^t\ ,\ -2e^{-2t}\rangle \Rightarrow \vec{r}'(0) = \langle 1\ ,\ -2\rangle$

(b). $x(t) = e^t$ and $y(t) = e^{-2t}$, we have $y = x^{-2}\ ,\ x > 0$.

The graph of the curve is shown below

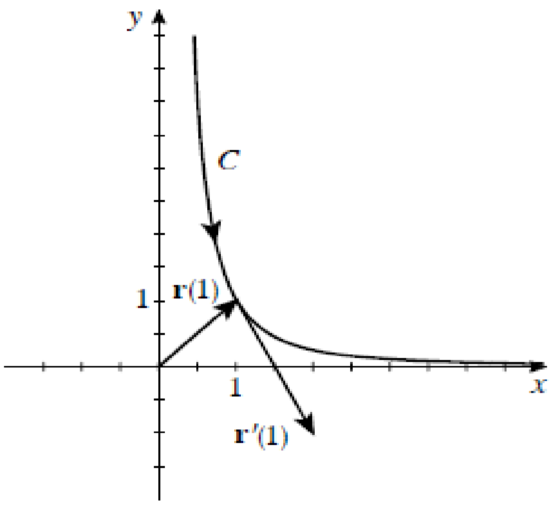

Figure 1: Graph of $y = x^{-2}$

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(20). Find the unit tangent vector $\vec{T}(t)$ at the point corresponding to given value of parameter t.

$$\boldsymbol{\vec{r}(t) = t\sin t\ \hat{i} + t\cos t\ \hat{j} + t\ \hat{k} \quad ; \quad t = \frac{\pi}{2}}$$

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○ Sol :

$$\vec{r}'(t) = (\sin t + t\cos t)\hat{i} + (\cos t - t\sin t)\hat{j} + \hat{k}$$

$$\therefore \quad \vec{r}'\left(\frac{\pi}{2}\right) = \hat{i} - \frac{\pi}{2}\hat{j} + \hat{k} \ \text{ and } \ \left|\vec{r}'\left(\frac{\pi}{2}\right)\right| = \sqrt{1 + \frac{\pi^2}{4} + 1} = \sqrt{2 + \frac{\pi^2}{4}} = \frac{\sqrt{8+\pi^2}}{2}$$

Hence the tangent vector $\vec{T}(\frac{\pi}{2})$ is

$$\vec{T}\left(\frac{\pi}{2}\right) = \frac{\vec{r}'\left(\frac{\pi}{2}\right)}{|\vec{r}'\left(\frac{\pi}{2}\right)|} = \frac{2}{\sqrt{8+\pi^2}}\left(\hat{i} - \frac{\pi}{2}\hat{j} + \hat{k}\right)$$

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(30). $\displaystyle \boldsymbol{\int_1^2 \left[\sqrt{t-1}\ \hat{i} + \frac{1}{\sqrt{t}}\ \hat{j} + (2t-1)^5\hat{k}\right] dt} = ?$

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○ Sol :

$$\int_1^2 \left[\sqrt{t-1}\ \hat{i} + \frac{1}{\sqrt{t}}\ \hat{j} + (2t-1)^5\hat{k}\right] dt = \left[\frac{2}{3}(t-1)^{3/2}\ \hat{i} + 2t^{1/2}\ \hat{j} + \frac{1}{12}(2t-1)^6\ \hat{k}\right]_1^2$$

$$= \left(\frac{2}{3}\ \hat{i} + 2\sqrt{2}\ \hat{j} + \frac{243}{4}\ \hat{k}\right) - \left(2\hat{j} + \frac{1}{12}\ \hat{k}\right) = \frac{2}{3}\ \hat{i} + 2(\sqrt{2}-1)\hat{j} + \frac{182}{3}\ \hat{k}$$
