Homework-8

Exercises of Section(10.3 , 11-2 , 11-3)

May 5, 2018

- **(Section 10.3)**

(4). Find the slope of the tangent line to the curve at the point corresponding to the value of the parameter

$$\boldsymbol{x = e^{2t}\ ,\ y = \ln t \quad ; \quad t = 1}$$

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∘ Sol :

The slope of the tangent line at $t = 1$ is

$$m = \frac{y'(t)}{x'(t)}\Big|_{t=1} = \frac{\frac{1}{t}}{2e^{2t}}\Big|_{t=1} = \frac{1}{2e^2}$$

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(8) Find an equation of the tangent line to the curve at the point corresponding to the value of the parameter

$$\boldsymbol{x = \theta\cos\theta,\ y = \theta\sin\theta \quad ; \quad \theta = \frac{\pi}{2}}$$

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∘ Sol :

The slope of the tangent line at $\theta = \frac{\pi}{2}$ is

$$m = \frac{y'(\theta)}{x'(\theta)}\Big|_{\theta=\frac{\pi}{2}} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\Big|_{\theta=\frac{\pi}{2}} = -\frac{2}{\pi}$$

The tangent point is

$$x = \theta\cos\theta\Big|_{\theta=\frac{\pi}{2}} = 0 \quad ; \quad y = \theta\sin\theta\Big|_{\theta=\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence the equation of the tangent line is

$$y - \frac{\pi}{2} = -\frac{2}{\pi}x \ \Rightarrow\ y = -\frac{2}{\pi}x + \frac{\pi}{2}$$

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(18) Find dy/dx and d^2y/dx^2

$$\boldsymbol{x = t^3 - t \ , \ y = t^3 + 2t^2}$$

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∘ Sol :

$$dx/dt = 3t^2 - 1 \quad ; \quad dy/dt = 3t^2 + 4t \ \Rightarrow \ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2+4t}{3t^2-1}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \left(\frac{1}{3t^2-1}\right)\frac{d}{dt}\left(\frac{3t^2+4t}{3t^2-1}\right) \\ &= \left(\frac{1}{3t^2-1}\right)\cdot\left(\frac{(6t+4)(3t^2-1)-6t(3t^2+4t)}{(3t^2-1)^2}\right) = \frac{-12t^2-6t-4}{(3t^2-1)^3} = -\frac{2(6t^2+3t+2)}{(3t^2-1)^3}\end{aligned}$$

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(21) Find dy/dx and d^2y/dx^2

$$\boldsymbol{x = \theta + \cos\theta \ , \ y = \theta - \sin\theta}$$

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∘ Sol :

$$dx/d\theta = 1 - \sin\theta \quad ; \quad dy/d\theta = 1 - \cos\theta \ \Rightarrow \ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1-\cos\theta}{1-\sin\theta}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \left(\frac{1}{1-\sin\theta}\right)\frac{d}{d\theta}\left(\frac{1-\cos\theta}{1-\sin\theta}\right) \\ &= \left(\frac{1}{1-\sin\theta}\right)\left(\frac{\sin\theta(1-\sin\theta)+\cos\theta(1-\cos\theta)}{(1-\sin\theta)^2}\right) = \frac{\sin\theta+\cos\theta-1}{(1-\sin\theta)^3}\end{aligned}$$

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(28) Find dy/dx and d^2y/dx^2 if

$$\boldsymbol{x = \int_1^t \frac{\sin u}{u} du \quad ; \quad y = \int_2^{\ln t} e^u du}$$

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∘ Sol :

$$\frac{dx}{dt} = \frac{d}{dt}\int_1^t \frac{\sin u}{u}du = \frac{\sin t}{t} \quad ; \quad \frac{dy}{dt} = \frac{d}{dt}\int_2^{\ln t} e^u du = e^{\ln t}\cdot\frac{d}{dt}\ln t = t\cdot\frac{1}{t} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t}{\sin t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{t}{\sin t}\cdot\frac{d}{dt}\left(\frac{t}{\sin t}\right) = \frac{t}{\sin t}\cdot\frac{\sin t - t\cos t}{\sin^2 t} = \frac{t(\sin t - t\cos t)}{\sin^3 t}$$

- **(Section 11.3)**

(42) Find $\text{proj}_{\vec{a}}\vec{b}$ and $\text{proj}_{\vec{b}}\vec{a}$

$$\boldsymbol{\vec{a} = \langle 1, 2, 0 \rangle \quad ; \quad \vec{b} = \langle -3, 0, -4 \rangle}$$

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○ Sol :

Given that $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = -3\hat{i} - 4\hat{k}$ we have

$$\vec{a} \cdot \vec{a} = 5 \quad ; \quad \vec{b} \cdot \vec{b} = 25 \quad ; \quad \vec{a} \cdot \vec{b} = -3$$

$$\Rightarrow \text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \right) \vec{a} = \frac{-3}{5}(\hat{i} + 2\hat{j}) = \frac{-3}{5}\hat{i} - \frac{6}{5}\hat{j}$$

$$\Rightarrow \text{proj}_{\vec{b}}\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \right) \vec{b} = \frac{-3}{25}(-3\hat{i} - 4\hat{k}) = \frac{9}{25}\hat{i} + \frac{12}{25}\hat{k}$$

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(46) Write $\vec{b}$ as the sum of a vector parallel to $\vec{a}$ and a vector perpendicular to $\vec{a}$

$$\boldsymbol{\vec{a} = -\hat{i} + 2\hat{j} \quad ; \quad \vec{b} = 2\hat{i} + 3\hat{j}}$$

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○ Sol :

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \right) \vec{a} = \frac{4}{5}(-\hat{i} + 2\hat{j}) = -\frac{4}{5}\hat{i} + \frac{8}{5}\hat{j}$$

$$\vec{b} - \text{proj}_{\vec{a}}\vec{b} = \frac{14}{5}\hat{i} + \frac{7}{5}\hat{j}$$

Hence we have

$$\vec{b} = \underbrace{\text{proj}_{\vec{a}}\vec{b}}_{\parallel} + \underbrace{(\vec{b} - \text{proj}_{\vec{a}}\vec{b})}_{\perp} = \left(-\frac{4}{5}\hat{i} + \frac{8}{5}\hat{j} \right) + \left(\frac{14}{5}\hat{i} + \frac{7}{5}\hat{j} \right)$$

- **(Section 11.4)**

(18) Find the area of the triangular with the given vertices

$$\boldsymbol{P(1,1,1)\ ,\ Q(1,2,1)\ ,\ R(2,2,3)}$$

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∘ Sol :

$$\overrightarrow{PQ} = (0,1,0) \quad ; \quad \overrightarrow{PR} = (1,1,2)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2\hat{i} - \hat{k}$$

Hence the area of $\triangle PQR$ is

$$\frac{1}{2}\left|2\hat{i} - \hat{k}\right| = \frac{1}{2} \cdot \sqrt{4+1} = \frac{\sqrt{5}}{2}$$

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(27) Find the volume of the parallelepiped determined by the vectors

$$\boldsymbol{\vec{a} = \hat{i} + \hat{j}\ ,\ \vec{b} = \hat{j} - 2\hat{k}\ ,\ \vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}}$$

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∘ Sol :

From the Thm.4 and Eq.(6) in Sec.(11.4) in text book(p.941-p.942), we have

$$Volume = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & 3 \end{vmatrix} = |3 - 2 + 4| = 5$$

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(44) Prove that

$$\boldsymbol{(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}}$$

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∘ Sol : Using the formula(7),(8) in Thm.3 in text book (p.940), we have

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= [(\vec{a} \times \vec{b}) \times \vec{c}] \cdot \vec{d} = [-\vec{c} \times (\vec{a} \times \vec{b})] \cdot \vec{d} \\ &= -[(\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}] \cdot \vec{d} \\ &= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d}) \\ &= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix} \end{aligned}$$
