Homework-7

Exercises of Section(9.8 , 10.2)

2018.04.17

- **(Section 9.8)**

(7). Find the Taylor series of f at c. Then find the radius of convergence of the series.

$$\boldsymbol{f(x) = \cos x \ , \ c = -\frac{\pi}{6}}$$

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∘ Sol :

$$f'(x) = -\sin x \quad ; \quad f''(x) = -\cos x \quad ; \quad f^{(3)}(x) = \sin x \quad ; \quad f^{(4)}(x) = \cos x \quad \cdots$$
$$f'(c) = \frac{1}{2} \quad ; \quad f''(c) = -\frac{\sqrt{3}}{2} \quad ; \quad f^{(3)}(c) = -\frac{1}{2} \quad ; \quad f^{(4)}(c) = \frac{\sqrt{3}}{2} \cdots$$

Hence the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}\left(x+\frac{\pi}{6}\right)^n = f(c) + f'(c)\left(x+\frac{\pi}{6}\right) + \frac{f''(c)}{2}\left(x+\frac{\pi}{6}\right)^2 + \frac{f^{(3)}(c)}{3!}\left(x+\frac{\pi}{6}\right)^3 + \cdots$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x+\frac{\pi}{6}\right) - \frac{\frac{\sqrt{3}}{2}}{2!}\left(x+\frac{\pi}{6}\right)^2 - \frac{\frac{1}{2}}{3!}\left(x+\frac{\pi}{6}\right)^3 + \frac{\frac{\sqrt{3}}{2}}{4!}\left(x+\frac{\pi}{6}\right)^4 \cdots$$
$$= \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x+\frac{\pi}{6}\right)^{2n} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x+\frac{\pi}{6}\right)^{2n+1}$$

Consider the series with even powers

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(2n+2)!}\left(x+\frac{\pi}{6}\right)^{2n+2} \cdot \frac{(2n)!}{(-1)^n\left(x+\frac{\pi}{6}\right)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{-(x+\frac{\pi}{6})^2}{(2n+1)(2n+2)}\right| = 0 \text{ for } x \in \mathbb{R}$$

Similarly, with odd powers we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(2n+3)!}\left(x+\frac{\pi}{6}\right)^{2n+3} \cdot \frac{(2n+1)!}{(-1)^n\left(x+\frac{\pi}{6}\right)^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{-\left(x+\frac{\pi}{6}\right)^2}{(2n+2)(2n+3)}\right| = 0 \text{ for } x \in \mathbb{R}$$

Hence the radius of convergence is ∞.

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(22) Use the power series representation of functions in this section to find the Taylor series of f at c and the radius of convergence.

$$f(x) = \sin^2 x \ , \ c = 0$$

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∘ Sol :

Using the Table-1 in the p799 in the text book, we have the Maclaurin Series of $\cos x$

$$\begin{aligned}
\Rightarrow \ f(x) = \sin^2 x &= \frac{1}{2}(1-\cos(2x)) = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n} \\
&= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n}\right) \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}}{(2n)!}\ x^{2n}
\end{aligned}$$

And the radius of convergence is ∞.

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(57). Find the sum of the given series.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

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∘ Sol :

Using the Table-1 in the p800 in text book, we know that

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Thus

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = \ln(1+x)\Big|_{x=1} = \ln 2$$

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(63). Evaluate

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

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∘ Sol :

Using the Table-1 in the p799 in the text book, we have the Maclaurin Series of $\sin x$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots$$

Hence

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0}\left(\frac{1}{120} - \frac{1}{7!}x^2 + \frac{1}{9!}x^4 - \cdots\right) = \frac{1}{120}$$

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(68).

(a) Find a power series representation of

$$f(x) = \sqrt[3]{1+x^2}$$

(b) Use the result of part(a) to find $f^{(6)}(0)$

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∘ Sol :

(a) Using the Binomial Series (p198 in text book)

$$f(x) = \sqrt[3]{1+x^2} = (1+x^2)^{1/3} = 1 + \frac{1}{3}\ x^2 + \frac{\frac{1}{3}\cdot(-\frac{2}{3})}{2!}\ x^4 + \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}}{3!}\ x^6 + \cdots$$

(b) Hence $f^{(6)}(0)$ is the coefficient of x^6 times 6!

$$f^{(6)}(0) = \left(\frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}}{3!}\right)\cdot 6! = \frac{10}{27}\cdot 4\cdot 5\cdot 6 = \frac{400}{9}$$

- **(Section 10.2)**

Find the rectangular equation of the given parametric equation and sketch the curve C and indicate it orentation.

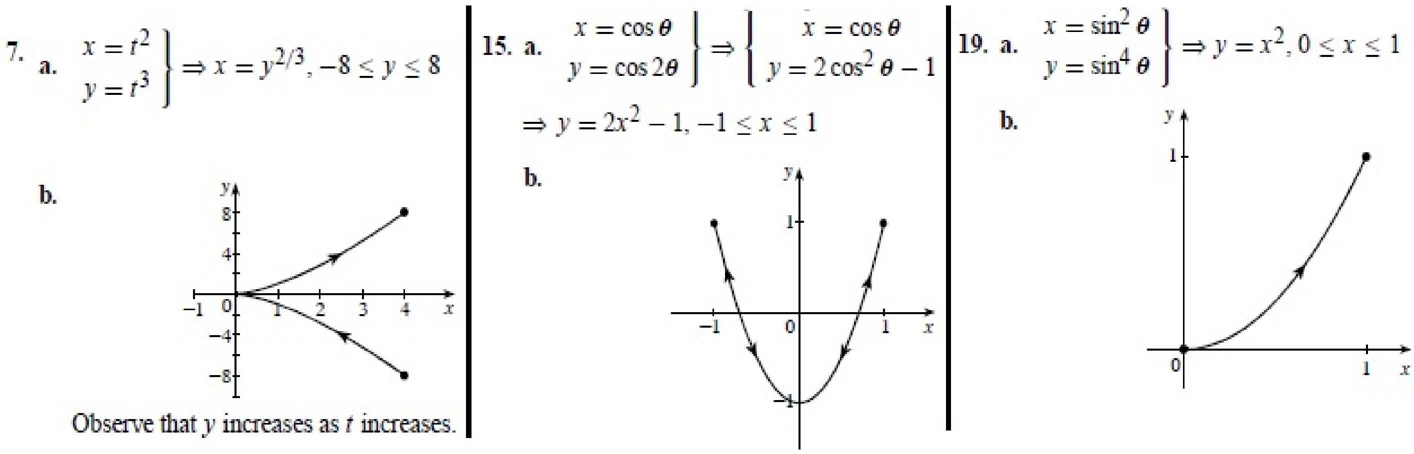

Figure 1: Exercise(7),(15) and (19)

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(39). Show that

$$x = a\sec t + h \quad ; \quad y = b\tan t + k \quad ; \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

are parametric equations of a hyperbola with center at (h,k) and transverse and conjugate axes of length $2a$ and $2b$.

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- Sol :

$$\sec t = \frac{x-h}{a} \quad ; \quad \tan t = \frac{y-k}{b} \Rightarrow \sec^2 t - \tan^2 t = \left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = 1 \tag{1}$$

Hence it is a equations of a hyperbola with center at (h,k) and transverse and conjugate axes of length $2a$ and $2b$.
