- ( 9.6 )

**4.** $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$. We use the Ratio Test on $a_n = \frac{(-1)^n 2^n}{n!}$, obtaining

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right] = \lim_{n\to\infty}\frac{2}{n+1} = 0,$$ so the series converges absolutely.

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**12.** $\sum_{n=1}^{\infty} \frac{\cos(n+1)}{n\sqrt{n}}$. Observe that $\left|\frac{\cos(n+1)}{n\sqrt{n}}\right| \le \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent $p$-series with $p = \frac{3}{2} > 1$, we conclude by the Comparison Test that the given series is absolutely convergent.

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**24.** $\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^n$. We use the Root Test: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\left(\frac{\ln n}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{\ln n}{n} = 0$, so the series converges absolutely.

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**34.** $\sum_{n=1}^{\infty}\frac{(n!)^2}{(3n)!}$. We use the Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left\{\frac{[(n+1)!]^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}\right\} = \lim_{n\to\infty}\frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)} = 0.$
Thus, the series converges absolutely.

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**42.** Consider $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!}\right] = \lim_{n\to\infty}2\left(\frac{n}{n+1}\right)^n = 2\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = 2\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1}$$
$$= \frac{2}{e} < 1,$$

so the series converges by the Ratio Test. Therefore, $\lim_{n\to\infty}\frac{2^n n!}{n^n} = 0$, which is the desired result.

- (9.7)

**6.** Let $u_n = \frac{(-1)^n x^n}{n \cdot 3^n}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)|x| = \frac{1}{3}|x| < 1 \Leftrightarrow$ $|x| < 3$, so $R = 3$ and the the series converges for $-3 < x < 3$. At $x = -3$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^n(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which diverges, and at $x = 3$ it is $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges. Thus, the interval of convergence is $(-3, 3]$.

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**8.** Let $u_n = \frac{n!x^n}{(2n)!}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n}\right| = \lim_{n\to\infty}\frac{|x|}{2(2n+1)} = 0$ for any real $x$, so the radius of convergence is $\infty$ and the interval of convergence is $(-\infty, \infty)$.

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**10.** Let $u_n = (x\ln n)^n = (\ln n)^n x^n$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{[\ln(n+1)]^{n+1} x^{n+1}}{(\ln n)^n x^n}\right| = \lim_{n\to\infty}\frac{[\ln(n+1)]^{n+1}}{(\ln n)^n}|x| = \infty$ for $x \neq 0$, so $R = 0$ and the series converges only at $x = 0$.

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**38.** $f(x) = \sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2 3^n}$. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)^2|x-2| = \frac{1}{3}|x-2| < 1$ $\Rightarrow |x-2| < 3$, so the series converges on $(-1, 5)$. At $x = -1$ the series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges, and at $x = 5$ it is $\sum_{n=1}^{\infty}\frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which also converges. Therefore, the interval of convergence is $[-1, 5]$. $f'(x) = \sum_{n=1}^{\infty}\frac{n(x-2)^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(x-2)^{n-1}}{n3^n}$ and $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^n}{(n+1)\,3^{n+1}} \cdot \frac{n3^n}{(x-2)^{n-1}}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)|x-2| = \frac{1}{3}|x-2| < 1 \Rightarrow |x-2| < 3 \Rightarrow f'$ converges on $(-1, 5)$. At $x = -1$ the series is $f'(-1) = \sum_{n=1}^{\infty}\frac{n(-3)^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n}$, which converges, and at $x = 5$ it is $f'(5) = \sum_{n=1}^{\infty}\frac{n3^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{1}{3n}$, which diverges. The interval of convergence is thus $[-1, 5)$.

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**39.** $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ and

$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^n}{n+1} \cdot \frac{n}{x^{n-1}} \right| = \lim_{n\to\infty} \left( \frac{n}{n+1} \right) |x| = |x|$, so $f'$ converges on $(-1, 1)$. If $x = -1$ the series is $f'(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which converges, and at $x = 1$ it is $f'(1) = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Thus, $f'$ has interval of convergence $[-1, 1)$. $f''(x) = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n}$ and $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{nx^{n-1}}{n+1} \cdot \frac{n}{(n-1)x^{n-2}} \right| = \lim_{n\to\infty} \left( \frac{n^2}{n^2-1} \right) |x| = |x|$, so $f''$ converges on $(-1, 1)$. At $x = -1$ we have $f''(-1) = \sum_{n=2}^{\infty} \frac{(n-1)(-1)^{n-2}}{n}$, and since $\lim_{n\to\infty} \left( \frac{n-1}{n} \right) = 1$, $\lim_{n\to\infty} \frac{(n-1)(-1)^{n-2}}{n}$ does not exist, and the Divergence Test shows that $f''(-1)$ is divergent. Similarly, $f''(1) = \sum_{n=2}^{\infty} \left( \frac{n-1}{n} \right)$ is divergent, so the interval of convergence of $f''$ is $(-1, 1)$. It is easy to see that the interval of convergence of $f$ is $[-1, 1]$.