Homework-2

Exercises of Section(7.3 - 7.4)

2018.03.13

- **(Section 7.3)**

Evaluate the integral using an appropriate trigonometric substitution.

- **(16).**

$$\int \frac{x^2}{\sqrt{3-x^2}}\, dx$$

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○ Sol :

Let $x = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta d\theta$

$$\int \frac{x^2}{\sqrt{3-x^2}}\, dx = \int \frac{3\sin^2\theta}{\sqrt{3}\cos\theta}\sqrt{3}\cos\theta\ d\theta$$
$$= 3\int \sin^2\theta d\theta = \frac{3}{2}\int 1-\cos 2\theta\ d\theta = \frac{3}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

Since

$$x = \sqrt{3}\sin\theta \ \Rightarrow\ \theta = \sin^{-1}\left(\frac{\sqrt{3}x}{3}\right) \quad ; \quad \sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{x}{\sqrt{3}}\cdot\frac{\sqrt{3-x^2}}{\sqrt{3}} = \frac{2}{3}x\sqrt{3-x^2}$$

Hence

$$\int \frac{x^2}{\sqrt{3-x^2}}\, dx = \frac{3}{2}\sin^{-1}\left(\frac{\sqrt{3}x}{3}\right) - \frac{1}{2}x\sqrt{3-x^2} + C$$

- **(20).**

$$\int \frac{1}{(9+x^2)^2}\, dx$$

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○ Sol :

Let $x = 3\tan\theta$, $dx = 3\sec^2\theta d\theta$

$$\int \frac{1}{(9+x^2)^2}\ dx = \int \frac{3\sec^2\theta}{81\sec^4\theta}\ d\theta = \frac{1}{27}\int \frac{1}{\sec^2\theta} d\theta$$
$$= \frac{1}{27}\int \cos^2\theta d\theta = \frac{1}{54}\int 1+\cos 2\theta\ d\theta = \frac{1}{54}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

Since

$$x = 3\tan\theta \ \Rightarrow\ \theta = \tan^{-1}\frac{x}{3} \quad ; \quad \sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{x}{\sqrt{x^2+9}}\cdot\frac{3}{\sqrt{x^2+9}} = \frac{6x}{x^2+9}$$

Hence

$$\int \frac{1}{(9+x^2)^2}\ dx = \frac{1}{54}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{18(x^2+9)} + C$$

- **(22).**

$$\boldsymbol{\int_2^4 \frac{\sqrt{x^2-4}}{x^4}\ dx}$$

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○ Sol :

Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta d\theta$

$$\int \frac{\sqrt{x^2-4}}{x^4}\ dx = \int \frac{2\tan\theta}{16\sec^4\theta}\cdot 2\sec\theta\tan\theta d\theta = \frac{1}{4}\int \frac{\tan^2\theta}{\sec^3\theta}\ d\theta$$
$$= \frac{1}{4}\int \sin^2\theta\cos\theta\ d\theta = \frac{1}{12}\sin^3\theta + C$$

Since

$$x = 2\sec\theta \ \Rightarrow\ \sin\theta = \frac{\sqrt{x^2-4}}{x} \ \Rightarrow\ \sin^3\theta = \frac{(x^2-4)^{3/2}}{x^3}$$

Hence

$$\int_2^4 \frac{\sqrt{x^2-4}}{x^4}\ dx = \left.\frac{(x^2-4)^{3/2}}{12x^3}\right|_2^4 = \frac{(2^2\cdot 3)^{3/2}}{12\cdot 64} = \frac{8}{12\cdot 64}\cdot 3\sqrt{3} = \frac{\sqrt{3}}{32}$$

- **(30).**

$$\int \frac{\boldsymbol{t^2}}{\sqrt{\boldsymbol{4t - t^2}}}\ \boldsymbol{dt}$$

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◦ Sol :

$$4t - t^2 = -(t^2 - 4t + 4 - 4) = -(t-2)^2 + 4$$

$$\Rightarrow \int \frac{t^2}{\sqrt{4t - t^2}}\ dt = \int \frac{t^2}{\sqrt{4 - (t-2)^2}}\ dt$$

Let $t - 2 = 2\sin\theta$, $dt = 2\cos\theta\ d\theta$

$$\int \frac{t^2}{\sqrt{4t - t^2}}\ dt = \int \frac{(2\sin\theta + 2)^2}{2\cos\theta}\ 2\cos\theta\ d\theta = \int 4\sin^2\theta + 8\sin\theta + 4\ d\theta$$
$$= 4\int \frac{1 - \cos 2\theta}{2}\ d\theta - 8\cos\theta + 4\theta + C_1 = 6\theta - \sin 2\theta - 8\cos\theta + C$$

Since

$$t - 2 = 2\sin\theta \ \Rightarrow\ \theta = \sin^{-1}\left(\frac{t-2}{2}\right) \quad ; \quad \sin\theta = \frac{t-2}{2} \quad ; \quad \cos\theta = \frac{\sqrt{4t - t^2}}{2}$$
$$\sin 2\theta = 2 \cdot \frac{t-2}{2} \cdot \frac{\sqrt{4t-t^2}}{2} = \frac{(t-2)\sqrt{4t-t^2}}{2}$$

Hence

$$\begin{aligned}\int \frac{t^2}{\sqrt{4t - t^2}}\ dt &= 6\sin^{-1}\left(\frac{t-2}{2}\right) - \frac{(t-2)\sqrt{4t-t^2}}{2} - 4\sqrt{4t - t^2} + C \\ &= 6\sin^{-1}\left(\frac{t-2}{2}\right) - \left(\frac{t-2+8}{2}\right)\sqrt{4t-t^2} + C \\ &= 6\sin^{-1}\left(\frac{t-2}{2}\right) - \left(\frac{t+6}{2}\right)\sqrt{4t-t^2} + C\end{aligned}$$

- **(56). Evaluate**

$$\int_0^{\frac{\pi}{4}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \quad ; \quad a > 0\ ,\ b > 0$$

Hint : Use the substitution $u = \tan x$

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◦ Sol :

$$u = \tan x \ \Rightarrow\ du = \sec^2 x\ dx \quad ; \quad \sin x = \frac{u}{\sqrt{1+u^2}} \quad ; \quad \cos x = \frac{1}{\sqrt{1+u^2}}$$

Then

$$a^2\cos^2 x + b^2\sin^2 x = \frac{a^2}{1+u^2} + \frac{b^2u^2}{1+u^2} = \frac{a^2+b^2u^2}{1+u^2} \quad ; \quad dx = \cos^2 x\ du = \frac{1}{1+u^2}\ du$$

Hence

$$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int \frac{1}{a^2+b^2u^2}\ du = \frac{1}{a^2}\int \frac{du}{1+\left(\frac{bu}{a}\right)^2}$$

Let $\frac{bu}{a} = y$, $\frac{b}{a}\ du = dy$, $du = (a/b)dy$

$$\Rightarrow \int \frac{du}{1+\left(\frac{bu}{a}\right)^2} = \int \frac{(a/b)dy}{1+y^2} = \frac{a}{b}\int \frac{1}{1+y^2}\ dy = \frac{a}{b}\tan^{-1} y + C = \frac{a}{b}\tan^{-1}\left(\frac{bu}{a}\right) + C$$

And

$$u = \tan x\ ,\ \text{then}\ \ x = \frac{\pi}{4} \ \Rightarrow\ u = 1 \quad ; \quad x = 0 \ \Rightarrow\ u = 0$$

Thus

$$\Rightarrow \int_0^{\pi/4} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{a^2}\cdot\frac{a}{b}\tan^{-1}\left(\frac{bu}{a}\right)\Bigg|_0^1 = \frac{1}{ab}\tan^{-1}\left(\frac{bu}{a}\right)\Bigg|_0^1 = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)$$

- **(Section 7.4)**

Evaluate the integral.

- **(12).**

$$\int \frac{1}{4x^2-9}\,dx$$

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∘ Sol :

$$\frac{1}{4x^2-9} = \frac{1}{(2x-3)(2x+3)} = \frac{A}{2x+3} + \frac{B}{2x-3}$$
$$\Rightarrow\ 2Bx+3B+2Ax-3A=1\ \Rightarrow\ B+A=0\quad;\quad B-A=\frac{1}{3}\ \Rightarrow\ A=-\frac{1}{6}\quad;\quad B=\frac{1}{6}$$

Hence

$$\int \frac{1}{4x^2-9}\,dx = -\frac{1}{6}\int\left(\frac{1}{2x+3}-\frac{1}{2x-3}\right)dx = -\frac{1}{12}\ln\left|\frac{2x+3}{2x-3}\right| + C = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$$

- **(14).**

$$\int_0^1 \frac{2u+3}{u^2+4u+3}\,du$$

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∘ Sol :

$$\frac{2u+3}{u^2+4u+3} = \frac{2u+3}{(u+1)(u+3)} = \frac{A}{u+1} + \frac{B}{u+3}$$
$$\Rightarrow\ Au+3A+Bu+B=2u+3\ \Rightarrow\ A+B=2\quad;\quad 3A+B=3\ \Rightarrow\ A=\frac{1}{2}\quad;\quad B=\frac{3}{2}$$

Hence

$$\int \frac{2u+3}{u^2+4u+3}\,du = \frac{1}{2}\int\frac{du}{u+1} + \frac{3}{2}\int\frac{du}{u+3} = \frac{1}{2}\ln|u+1| + \frac{3}{2}\ln|u+3| + C$$

$$\Rightarrow\ \int_0^1 \frac{2u+3}{u^2+4u+3}\,du = \frac{1}{2}\ln|u+1|\Big|_0^1 + \frac{3}{2}\ln|u+3|\Big|_0^1 = \frac{1}{2}\ln 2 + \frac{3}{2}\ln 4 - \frac{3}{2}\ln 3 = \frac{7}{2}\ln 2 - \frac{3}{2}\ln 3$$

- **(22).**

$$\boldsymbol{\int_2^4 \frac{3x-5}{(x-1)^2}\,dx}$$

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- Sol :

$$\frac{3x-5}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

So

$$A(x-1) + B = 3x - 5 \;\Rightarrow\; A = 3 \quad ; \quad B - A = -5 \;\Rightarrow\; B = -2$$

Then

$$\int \frac{3x-5}{(x-1)^2}\,dx = \int \left(\frac{3}{x-1} - \frac{2}{(x-1)^2}\right) dx = 3\ln|x-1| + 2(x-1)^{-1} + C$$

Hence

$$\int_2^4 \frac{3x-5}{(x-1)^2}\,dx = 3\ln|x-1|\Big|_2^4 + 2(x-1)^{-1}\Big|_2^4 = 3\ln 3 + 2\cdot\frac{1}{3} - 2 = 3\ln 3 - \frac{4}{3}$$

- **(26).**

$$\int \frac{4x^2}{(x^2-4)^2}\,dx$$

◦ Sol :

$$\frac{4x^2}{(x^2-4)^2} = \frac{4x^2}{(x-2)^2(x+2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

$$\text{Let } f(x) = \frac{4x^2}{(x^2-4)^2}$$

$$\Rightarrow \ (x-2)^2 \cdot f(x)\Big|_{x=2} = B = \frac{4x^2}{(x+2)^2}\Big|_{x=2} = 1$$

$$\Rightarrow \ (x+2)^2 \cdot f(x)\Big|_{x=-2} = D = \frac{4x^2}{(x-2)^2}\Big|_{x=-2} = 1$$

Hence

$$\begin{aligned}
\frac{4x^2}{(x^2-4)^2} &= \frac{A}{(x-2)} + \frac{C}{(x+2)} + \frac{1}{(x-2)^2} + \frac{1}{(x+2)^2} \\
&= \frac{A(x-2)(x+2)^2 + B(x+2)(x-2)^2 + (x+2)^2 + (x-2)^2}{(x-2)^2(x+2)^2} \\
&= \frac{A(x^3+2x^2-4x-8) + C(x^3-2x^2-4x+8) + x^2+4x+4+x^2-4x+4}{(x^2-4)^2} \\
&= \frac{(A+C)x^3 + (2A-2C+2)x^2 + (-4A-4C)x - 8A + 8C + 8}{(x^2-4)^2}
\end{aligned}$$

We have

$$A + C = 0 \quad ; \quad A - C = 1 \ \Rightarrow \ A = \frac{1}{2} \quad ; \quad C = -\frac{1}{2}$$

Finally we have

$$\begin{aligned}
\int \frac{4x^2}{(x^2-4)^2}\,dx &= \int \left(\frac{1/2}{(x-2)} + \frac{1}{(x-2)^2} - \frac{1/2}{(x+2)} + \frac{1}{(x+2)^2} \right) dx \\
&= \frac{1}{2}\ln|x-2| - (x-2)^{-1} - \frac{1}{2}\ln|x+2| - (x+2)^{-1} + C \\
&= \frac{1}{2}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{x-2} - \frac{1}{x+2} + C \\
&= \frac{1}{2}\ln\left|\frac{x-2}{x+2}\right| - \frac{2x}{x^2-4} + C
\end{aligned}$$

- **(36).**

$$\int \boldsymbol{\frac{x^2+1}{x^3-1}\,dx}$$

◦ Sol :

$$\frac{x^2+1}{x^3-1} = \frac{x^2+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

So we have

$$\begin{aligned} & A(x^2+x+1) + (Bx+C)(x-1) = x^2+1 \\ \Rightarrow \quad & Ax^2 + Ax + A + Bx^2 + Cx - Bx - C = x^2+1 \\ \Rightarrow \quad & (A+B)x^2 + (A-B+C)x + (A-C) = x^2+1 \\ \Rightarrow \quad & A+B=1 \quad ; \quad A-B+C=0 \quad ; \quad A-C=1 \end{aligned}$$

Hence we get

$$A = \frac{2}{3} \quad ; \quad B = \frac{1}{3} \quad ; \quad C = -\frac{1}{3}$$

Then

$$\begin{aligned} \int \frac{x^2+1}{x^3-1}\,dx &= \int \left(\frac{2/3}{x-1} + \frac{\frac{1}{3}x - \frac{1}{3}}{x^2+x+1} \right) dx \\ &= \frac{2}{3}\ln|x-1| + \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx \\ &= \frac{2}{3}\ln|x-1| + \frac{1}{3}\left(\frac{1}{2}\int \frac{2x-2}{x^2+x+1}\,dx \right) \\ &= \frac{2}{3}\ln|x-1| + \frac{1}{6}\left(\int \frac{2x+1-3}{x^2+x+1}\,dx \right) \\ &= \frac{2}{3}\ln|x-1| + \frac{1}{6}\left(\int \frac{2x+1}{x^2+x+1}\,dx - 3\int \frac{1}{x^2+x+1}\,dx \right) \\ &= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx \qquad (1) \end{aligned}$$

Deal with the

$$\int \frac{1}{x^2+x+1}\,dx = ??$$

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = (x+1/2)^2 + 3/4$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+1/2)^2+3/4}\,dx$$

$$\because\ (x+\frac{1}{2})^2 + \frac{3}{4} = \frac{3}{4}\left(1 + \frac{4}{3}(x+\frac{1}{2})^2\right) = \frac{3}{4}\left(1 + \left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2\right)$$

$$\therefore \int \frac{1}{x^2+x+1}\ dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2}\ dx$$

Let $\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}} = y$, $\frac{2}{\sqrt{3}}dx = dy$, $dx = \frac{\sqrt{3}}{2}dy$

$$\Rightarrow \int \frac{1}{x^2+x+1}\ dx = \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{dy}{1+y^2} = \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C \tag{2}$$

Take Eq.(2) into Eq.(1) we have

$$\begin{aligned}\int \frac{x^2+1}{x^3-1}\ dx &= \frac{2}{3}\ln\left|x-1\right| + \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(2x+1)}{3}\right)+C \\ &= \frac{1}{6}\left[\ln\left((x-1)^4(x^2+x+1)\right) - 2\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}(2x+1)}{3}\right)\right]+C\end{aligned}$$