14.8 Change of Variables in Multiple Integrals

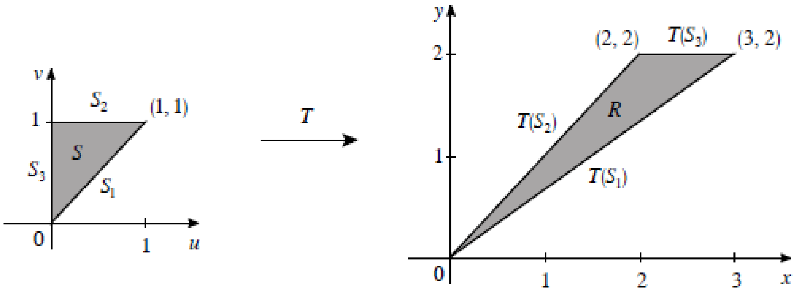
3. Here $T: x = u + 2v,\ y = 2v$.

On S_1: $v = u$, $0 \le u \le 1$. This is mapped onto $T(S_1)$: $y = \frac{2}{3}x$, $0 \le x \le 3$.

On S_2: $v = 1$, $0 \le u \le 1$. This is mapped onto $T(S_2)$: $y = 2$, $2 \le x \le 3$.

On S_3: $u = 0$, $0 \le v \le 1$. This is mapped onto $T(S_3)$: $y = x$, $0 \le x \le 2$.

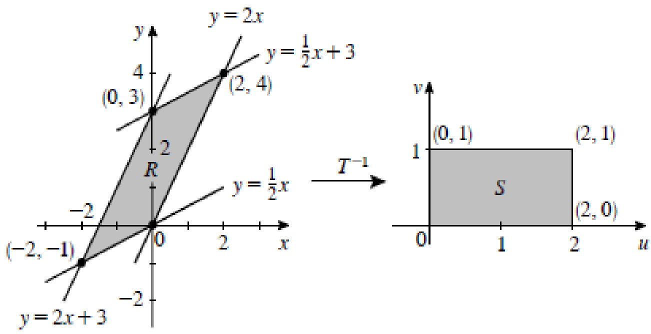
Thus, R is the triangular region shown.



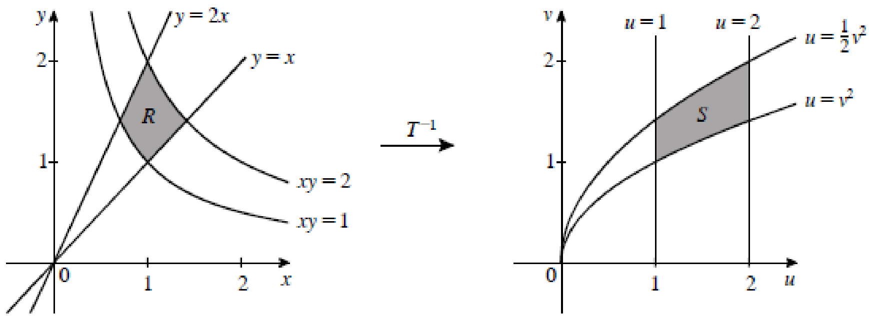
9. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos 2v & -2e^u \sin 2v \\ e^u \sin 2v & 2e^u \cos 2v \end{vmatrix} = 2e^{2u}\left(\cos^2 2v + \sin^2 2v\right) = 2e^{2u}$

14. To find T^{-1}, we solve the system of equations of $T: x = u - 2v,\ y = 2u - v$ for u and v, obtaining $T^{-1}: u = \frac{1}{3}(2y - x),\ v = \frac{1}{3}(y - 2x)$. Using this transformation, we obtain the region $S = T^{-1}(R)$. Next, we find

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = 3.$$



18. The transformation is $T: x = \dfrac{u}{v},\ y = v$. Then $xy = 1 \Rightarrow \dfrac{u}{v} \cdot v = 1 \Leftrightarrow u = 1$, $xy = 2 \Rightarrow u = 2$, $y = x \Rightarrow v = \dfrac{u}{v} \Rightarrow u = v^2$, and $y = 2x \Rightarrow u = \dfrac{v^2}{2}$.

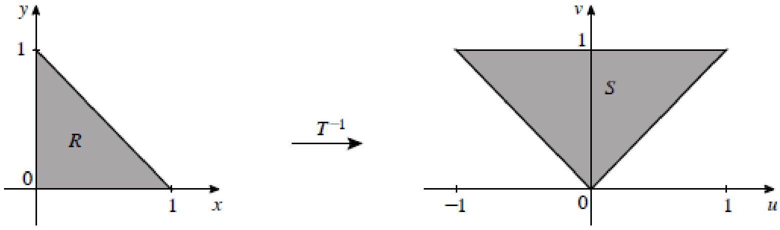


Next, we find $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, and so

$$\iint_R xy^2\, dA = \iint_S \frac{u}{v} \cdot v^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du = \int_1^2 \int_{\sqrt{u}}^{\sqrt{2u}} u\, dv\, du = \int_1^2 [uv]_{v=\sqrt{u}}^{v=\sqrt{2u}}\, du = \int_1^2 \left(\sqrt{2}u^{3/2} - u^{3/2}\right) du$$

$$= \left(\sqrt{2} - 1\right)\left(\tfrac{2}{5}\right) u^{5/2}\Big|_1^2 = \tfrac{2}{5}\left(9 - 5\sqrt{2}\right)$$

23. Let $T: u = x - y,\ v = x + y$. Then $T^{-1}: x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(v-u)$. The triangular region R is mapped onto the triangular region S.



Here $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$, so

$$\iint_R \exp\left(\frac{x-y}{x+y}\right) dA = \int_0^1 \int_{-v}^{v} e^{u/v} \left|\tfrac{1}{2}\right| du\,dv = \tfrac{1}{2}\int_0^1 \left[ve^{u/v}\right]_{u=-v}^{u=v} dv = \tfrac{1}{2}\int_0^1 v\left(e - e^{-1}\right) dv = \tfrac{1}{2}\left(e - e^{-1}\right)\left(\tfrac{1}{2}v^2\right)\Big|_0^1$$
$$= \frac{e^2-1}{4e}$$

14.2 Iterated Integrals

7. $\int_0^4 \int_0^{\sqrt{x}} 2xy\,dy\,dx = \int_0^4 \left[\int_0^{x^{1/2}} 2xy\,dy\right] dx = \int_0^4 \left[xy^2\right]_{y=0}^{y=x^{1/2}} dx = \int_0^4 x^2\,dx = \frac{1}{3}x^3\Big|_0^4 = \frac{64}{3}$

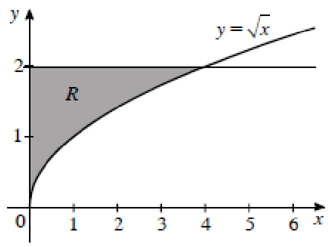
8. $\int_0^{1/2} \int_0^{\sqrt{1-x}} 2xy\,dy\,dx = \int_0^{1/2} \left[\int_0^{\sqrt{1-x}} 2xy\,dy\right] dx = \int_0^{1/2} \left[xy^2\right]_{y=0}^{y=\sqrt{1-x}} dx = \int_0^{1/2} x(1-x)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^{1/2} = \frac{1}{12}$

16. $\iint_R ye^{xy}\,dA = \int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 \left(e^y - 1\right) dy = \left(e^y - y\right)\Big|_0^1 = e - 2$

57. $\int_0^4 \int_{\sqrt{x}}^2 \sin y^3\,dy\,dx = \int_0^2 \int_0^{y^2} \sin y^3\,dx\,dy = \int_0^2 \left[x \sin y^3\right]_{x=0}^{x=y^2} dy$

$= \int_0^2 y^2 \sin y^3\,dy = -\frac{1}{3}\cos y^3\Big|_0^2$

$= \dfrac{1-\cos 8}{3}$



59. $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{\sqrt{x^3+1}}\,dx\,dy = \int_0^2 \int_0^{x^2} \frac{1}{\sqrt{x^3+1}}\,dy\,dx = \int_0^2 \left[\frac{y}{\sqrt{x^3+1}}\right]_{y=0}^{y=x^2} dx$

$\displaystyle = \int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{2}{3}\left(x^3+1\right)^{1/2}\Big|_0^2$

$= \frac{2}{3}(3-1) = \frac{4}{3}$