13.9 Lagrange Multipliers

6. $f(x,y)=x^2-y^2 \Rightarrow \nabla f(x,y)=2x\mathbf{i}-2y\mathbf{j}$ and $g(x,y)=x^2+y^2-1 \Rightarrow \nabla g(x,y)=2x\mathbf{i}+2y\mathbf{j}$, so $\nabla f=\lambda\nabla g \Rightarrow$ $2x\mathbf{i}-2y\mathbf{j}=2\lambda x\mathbf{i}+2\lambda y\mathbf{j}$, and we solve $\left.\begin{aligned} 2x&=2\lambda x & (1)\\ -2y&=2\lambda y & (2)\\ x^2+y^2&=1 & (3)\end{aligned}\right\}$ From (1) we see that $x=0$ or $\lambda=1$. If $x=0$, then (3) gives $y=\pm1$. If $\lambda=1$, then (2) gives $y=0$ and (3) then gives $x=\pm1$.

(x,y)	$(0,1)$	$(0,-1)$	$(-1,0)$	$(1,0)$
$f(x,y)$	-1	-1	1	1

From the table, we see that the minimum value of f is -1 and the maximum value of f is 1.

9. $f(x,y)=x^2+xy+y^2 \Rightarrow \nabla f(x,y)=(2x+y)\mathbf{i}+(x+2y)\mathbf{j}$ and $g(x,y)=x^2+y^2-8 \Rightarrow \nabla g(x,y)=2x\mathbf{i}+2y\mathbf{j}$, so $\nabla f=\lambda\nabla g \Rightarrow (2x+y)\mathbf{i}+(x+2y)\mathbf{j}=2\lambda x\mathbf{i}+2\lambda y\mathbf{j}$, and we solve $\left.\begin{aligned} 2x+y&=2\lambda x\\ x+2y&=2\lambda y\\ x^2+y^2&=8\end{aligned}\right\} \Rightarrow \left.\begin{aligned} x&=2y(\lambda-1)\\ y&=2x(\lambda-1)\end{aligned}\right\} \Rightarrow$ $y=\pm x$ (because $\lambda=1 \Rightarrow x=y=0$, violating the third equation). Substituting $y=\pm x$ into the third equation gives $x=\pm2$ and $y=\pm2$.

(x,y)	$(-2,-2)$	$(-2,2)$	$(2,-2)$	$(2,2)$
$f(x,y)$	12	4	4	12

From the table, we see that f has a minimum value of 4 and a maximum value of 12.

13. $f(x,y,z)=x+2y-2z \Rightarrow \nabla f(x,y,z)=\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $g(x,y,z)=x^2+2y^2+4z^2-1 \Rightarrow$ $\nabla g(x,y,z)=2x\mathbf{i}+4y\mathbf{j}+8z\mathbf{k}$, so $\nabla f=\lambda\nabla g \Rightarrow \mathbf{i}+2\mathbf{j}-2\mathbf{k}=2\lambda x\mathbf{i}+4\lambda y\mathbf{j}+8\lambda z\mathbf{k}$, and we solve $\left.\begin{aligned} 1&=2\lambda x\\ 2&=4\lambda y\\ -2&=8\lambda z\\ x^2+2y^2+4z^2&=1\end{aligned}\right\}$ We see that $x=\dfrac{1}{2\lambda}$, $y=\dfrac{1}{2\lambda}$, and $z=-\dfrac{1}{4\lambda}$, which we substitute into the fourth equation: $\left(\dfrac{1}{2\lambda}\right)^2+2\left(\dfrac{1}{2\lambda}\right)^2+4\left(-\dfrac{1}{4\lambda}\right)^2=1 \Rightarrow \lambda=\pm1$. If $\lambda=-1$, then $x=-\frac12$, $y=-\frac12$, and $z=\frac14$; if $\lambda=1$, then $x=\frac12$, $y=\frac12$, and $z=-\frac14$. We see that the minimum value of f is $f\left(-\frac12,-\frac12,\frac14\right)=-2$ and the maximum value of f is $f\left(\frac12,\frac12,-\frac14\right)=2$.

18. $f(x,y,z)=x+y+z \Rightarrow \nabla f(x,y,z)=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $g(x,y,z)=x^2+y^2-1 \Rightarrow \nabla g(x,y,z)=2x\mathbf{i}+2y\mathbf{j}$, and $h(x,y,z)=x+z-2 \Rightarrow \nabla h(x,y,z)=\mathbf{i}+\mathbf{k}$, so $\nabla f=\lambda\nabla g+\mu\nabla h$ along with the constraints $g(x,y,z)=0$ and $h(x,y,z)=0$ give the system $\left.\begin{aligned} 1&=2\lambda x+\mu & (1)\\ 1&=2\lambda y & (2)\\ 1&=\mu & (3)\\ x^2+y^2&=1 & (4)\\ x+z&=2 & (5)\end{aligned}\right\}$ Substituting (3) into (1) gives $2\lambda x=0$, but (2) implies that $\lambda\neq0$. So $x=0$, and therefore (4) gives $y=\pm1$ and (5) gives $z=2$. We consider the points $(0,-1,2)$ and $(0,1,2)$, and find that f has minimum value $f(0,-1,2)=1$ and maximum value $f(0,1,2)=3$.

22. $\left.\begin{aligned} f_x(x,y)&=\tfrac{\partial}{\partial x}\left(x^2y\right)=2xy=0\\ f_y(x,y)&=\tfrac{\partial}{\partial y}\left(x^2y\right)=x^2\;\;\,=0\end{aligned}\right\} \Rightarrow x=y=0$, so f has the critical point $(0,0)$ in the disk $D=\left\{(x,y)\mid 4x^2+y^2\le4\right\}$. Next, we use the method of Lagrange to find the critical points of f on the boundary of D. Writing $g(x,y)=4x^2+y^2-4=0$, we find $\nabla f(x,y)=2xy\mathbf{i}+x^2\mathbf{j}$ and $\nabla g(x,y)=8x\mathbf{i}+2y\mathbf{j}$. The equation $\nabla f=\lambda\nabla g$ and the constraint equation $g(x,y)=0$ give the system $\left.\begin{aligned} 2xy&=8\lambda x & (1)\\ x^2&=2\lambda y & (2)\\ 4x^2+y^2&=4 & (3)\end{aligned}\right\}$ Equation (1) gives $x=0$ or $\lambda=\frac14y$. If $x=0$, then (3) gives $y=\pm2$, and if $\lambda=\frac14y$, then (2) gives $x^2=2\left(\frac14y\right)y=\frac12y^2 \Rightarrow y^2=2x^2$. Substituting this into (3) gives $4x^2+2x^2=4 \Leftrightarrow x=\pm\frac{\sqrt6}{3}$, and the corresponding values of y are $\pm\frac{2\sqrt3}{3}$.

(x,y)	$(0,0)$	$(0,-2)$	$(0,2)$	$\left(-\frac{\sqrt6}{3},-\frac{2\sqrt3}{3}\right)$	$\left(-\frac{\sqrt6}{3},\frac{2\sqrt3}{3}\right)$	$\left(\frac{\sqrt6}{3},-\frac{2\sqrt3}{3}\right)$	$\left(\frac{\sqrt6}{3},\frac{2\sqrt3}{3}\right)$
$f(x,y)$	0	0	0	$-\frac{4\sqrt3}{9}$	$\frac{4\sqrt3}{9}$	$-\frac{4\sqrt3}{9}$	$\frac{4\sqrt3}{9}$

From the table, we see that f has a minimum value of $-\frac{4\sqrt3}{9}$ and a maximum value of $\frac{4\sqrt3}{9}$.

14.1 Double Integrals

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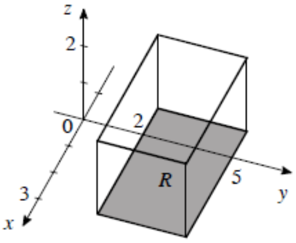
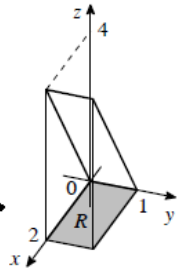
13. $\iint_R 2\,dA$ represents the volume of the rectangular box with base $R = [-1, 3] \times [2, 5]$ and height 2, so $\iint_R 2\,dA = 2\,[3 - (-1)]\,(5 - 2) = 24$.



14. $\iint_R 2x\,dA$ represents the volume of the solid shown. Its base is $R = [0, 2] \times [0, 1]$, so $\iint_R 2x\,dA = \frac{1}{2}\,(2)\,(1)\,(4) = 4$.



25. Observe that $0 < e^{-1} \leq e^{-x} \leq 1$ for all x in $[0, 1]$ and $0 < \cos y \leq 1$ for all $0 \leq y \leq 1$, so $0 \leq e^{-x}\cos y \leq 1$ for all $(x, y) \in R$. Thus, by Property 4 of Theorem 1, we have $\iint_R 0\,dA \leq \iint_R e^{-x}\cos y\,dA \leq \iint_R 1\,dA \Leftrightarrow 0 \leq \iint_R e^{-x}\cos y\,dA \leq 1$.

26. Observe that $0 \leq \sin(2x + 3y) \leq 1$ on $R = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$, so by Property 4 of Theorem 1, we have $\iint_R 0\,dA \leq \iint_R \sin(2x + 3y)\,dA \leq \iint_R 1\,dA = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.