(Section 13.7)

6. $F(x,y) = x^4 - x^2 + y^2 \Rightarrow \nabla F\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) = \left[\left(4x^3 - 2x\right)\mathbf{i} + 2y\mathbf{j}\right]_{\left(1/2, \sqrt{3}/4\right)} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ is normal to the level curve $F(x,y) = x^4 - x^2 + y^2 = 0$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$. So the slope of the required normal line is $m = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$ and an equation of the normal line is $y - \frac{\sqrt{3}}{4} = -\sqrt{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = -\sqrt{3}x + \frac{3\sqrt{3}}{4}$. The slope of the required tangent line is $m = -\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$, and so an equation of the tangent line is $y - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{12}$.

16. $F(x,y,z) = 2x^2 - y^2 + 3z^2 - 2 = 0 \Rightarrow \nabla F(2,-3,1) = (4x\mathbf{i} - 2y\mathbf{j} + 6z\mathbf{k})|_{(2,-3,1)} = 8\mathbf{i} + 6\mathbf{j} + 6\mathbf{k} = 2(4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$, so an equation of the tangent plane at $(2,-3,1)$ is $4(x-2) + 3(y+3) + 3(z-1) = 0 \Leftrightarrow 4x + 3y + 3z = 2$. Equations of the normal line passing through $(2,-3,1)$ are $\frac{x-2}{4} = \frac{y+3}{3} = \frac{z-1}{3}$.

26. $F(x,y,z) = e^x \sin \pi y - z = 0 \Rightarrow \nabla F(0,1,0) = \left(e^x \sin \pi y\, \mathbf{i} + \pi e^x \cos \pi y\, \mathbf{j} - \mathbf{k}\right)\big|_{(0,1,0)} = -\pi\mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $(0,1,0)$ is $-\pi(y-1) - z = 0 \Leftrightarrow \pi y + z = \pi$. Equations of the normal line passing through $(0,1,0)$ are $x = 0$, $\frac{y-1}{-\pi} = \frac{z}{-1} \Leftrightarrow x = 0$, $\frac{y-1}{\pi} = z$.

30. $F(x,y,z) = x\cos y - z = 0 \Rightarrow \nabla F\left(2, \frac{\pi}{3}, 1\right) = \left(\cos y\, \mathbf{i} - x \sin y\, \mathbf{j} - \mathbf{k}\right)|_{(2,\pi/3,1)} = \frac{1}{2}\mathbf{i}\text{-}\sqrt{3}\mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $\left(2, \frac{\pi}{3}, 1\right)$ is $\frac{1}{2}(x-2) - \sqrt{3}\left(y - \frac{\pi}{3}\right) - (z-1) = 0 \Leftrightarrow x - 2\sqrt{3}y - 2z = -\frac{2\pi\sqrt{3}}{3}$. Equations of the normal line passing through $\left(2, \frac{\pi}{3}, 1\right)$ are $\frac{x-2}{\frac{1}{2}} = \frac{y - \frac{\pi}{3}}{-\sqrt{3}} = \frac{z-1}{-1} \Leftrightarrow x - 2 = \frac{y - \frac{\pi}{3}}{-2\sqrt{3}} = \frac{z-1}{-2}$.

34. $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 \Rightarrow \nabla F(x_0,y_0,z_0) = \frac{2x_0}{a^2}\mathbf{i} + \frac{2y_0}{b^2}\mathbf{j} - \frac{2z_0}{c^2}\mathbf{k}$, so an equation of the tangent plane at (x_0,y_0,z_0) is $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0 \Leftrightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 0$. But (x_0,y_0,z_0) lies on the hyperboloid, so the expression in parentheses is equal to 1 and we have $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 1$, as was to be shown.

(Section 13.8)

13. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2 - 6x - x\sqrt{y} + y\right) = 2x - 6 - \sqrt{y} = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2 - 6x - xy^{1/2} + y\right) = -\dfrac{x}{2\sqrt{y}} + 1 = 0 \end{array}\right\}$ From the first equation, we see that $\sqrt{y} = 2x - 6$. Substituting this into the second equation gives $-x + 2(2x - 6) = 0 \Rightarrow x = 4$. Substituting this into the first equation gives $8 - 6 = \sqrt{y} \Rightarrow y = 4$, so the sole critical point of f is $(4, 4)$. Next, $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y) = 2\left(\dfrac{x}{4y^{3/2}}\right) - \left(-\dfrac{1}{2\sqrt{y}}\right)^2 = \dfrac{x}{2y^{3/2}} - \dfrac{1}{4y}$. Since $D(4,4) = \frac{4}{2(8)} - \frac{1}{4(4)} = \frac{3}{16} > 0$ and $f_{xx}(4,4) = 2 > 0$, the point $(4, 4)$ gives a relative minimum of f with value $f(4,4) = -12$.

16. $\left.\begin{array}{l} f_x(x,y) = \dfrac{\partial}{\partial x}\left(-\dfrac{4y}{x^2+y^2+1}\right) = \dfrac{8xy}{\left(x^2+y^2+1\right)^2} = 0 \\ f_y(x,y) = \dfrac{\partial}{\partial y}\left(-\dfrac{4y}{x^2+y^2+1}\right) = \dfrac{4\left(y^2 - x^2 - 1\right)}{\left(x^2+y^2+1\right)^2} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} 8xy = 0 \\ y^2 - x^2 - 1 = 0 \end{array}\right\}$ The first equation gives $x = 0$ or $y = 0$. Setting $x = 0$ in the second equation gives $y = \pm 1$; setting $y = 0$ gives no solution. Thus, f has critical points $(0, -1)$ and $(0, 1)$. Next, $f_{xx}(x,y) = \dfrac{8y\left(1 - 3x^2 + y^2\right)}{\left(x^2+y^2+1\right)^3}$, $f_{xy}(x,y) = \dfrac{8x\left(x^2 - 3y^2 + 1\right)}{\left(x^2+y^2+1\right)^3}$, and $f_{yy}(x,y) = \dfrac{8y\left(3x^2 - y^2 + 3\right)}{\left(x^2+y^2+1\right)^3}$.

At $(0,-1)$: $f_{xx}(0,-1) = -2$, $f_{xy}(0,-1) = 0$, and $f_{yy}(0,-1) = -2$, so $D(0,-1) = f_{xx}(0,-1)\, f_{yy}(0,-1) - f_{xy}^2(0,-1) = (-2)(-2) - 0^2 = 4 > 0$. Since $f_{xx}(0,-1) = -2 < 0$, we see that $(0,-1)$ gives a relative maximum of f with value $f(0,-1) = 2$.

At $(0,1)$: $f_{xx}(0,1) = 2$, $f_{xy}(0,1) = 0$, and $f_{yy}(0,1) = 2$, so $D(0,1) = f_{xx}(0,1)\, f_{yy}(0,1) - f_{xy}^2(0,1) = (2)(2) - 0^2 = 4 > 0$. Since $f_{xx}(0,1) = 2 > 0$, we see that $(0,1)$ gives a relative minimum of f with value $f(0,1) = -2$.

22. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}(\sin x + \sin y) = \cos x = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}(\sin x + \sin y) = \cos y = 0 \end{array}\right\} \Rightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ and $y = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Thus, f has the critical points $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$. Next, $f_{xx}(x,y) = -\sin x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = -\sin y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y) = \sin x \sin y$.

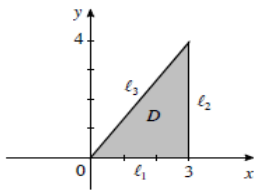
At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$: $D\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 1 > 0$ and $f_{xx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -1 < 0$, so f has the relative maximum value $f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 2$.

At $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$: $D\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) = -1 < 0 \Rightarrow f$ has the saddle point $\left(\frac{\pi}{2}, \frac{3\pi}{2}, 0\right)$.

At $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$: $D\left(\frac{3\pi}{2}, \frac{\pi}{2}\right) = -1 < 0 \Rightarrow f$ has the saddle point $\left(\frac{3\pi}{2}, \frac{\pi}{2}, 0\right)$.

At $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$: $D\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right) = 1 > 0$ and $f_{xx}\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right) = 1 > 0$, so f has the relative minimum value $f\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right) = -2$.

35. $f_x(x,y) = 3$ and $f_y(x,y) = 4$, so f has no critical point.



On ℓ_1, $x = x$ and $y = 0$, so $g(y) = f(x, 0) = 3x - 12$ for $0 \le x \le 3$. We see that f has an absolute minimum value of -12 and an absolute maximum value of -3 on ℓ_1.

On ℓ_2, $x = 3$ and $y = y$, so $h(y) = f(3, y) = 4y - 3$ for $0 \le y \le 4$. We see that f has an absolute minimum value of -3 and an absolute maximum value of 13 on ℓ_2.

On ℓ_3, $y = \frac{4}{3}x$, so $s(x) = f\left(x, \frac{4}{3}x\right) = 3x + 4\left(\frac{4}{3}x\right) - 12 = \frac{25}{3}x - 12$ for $0 \le x \le 3$. We see that f has an absolute minimum value of -12 and an absolute maximum value of 13 on ℓ_3.

From these calculations, we see that f has an absolute minimum value of -12 and an absolute maximum value of 13 on D.

43. We want to minimize $d^2 = f(x,y) = x^2 + y^2 + z^2 = x^2 + y^2 + xy - x + 4y + 21$. $\left.\begin{array}{l} f_x(x,y) = 2x + y - 1 = 0 \\ f_y(x,y) = 2y + x + 4 = 0 \end{array}\right\} \Rightarrow$ $x = 2$ and $y = -3$, so $(2, -3)$ is a critical point of f. $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y) = 2 \cdot 2 - 1^2 = 3 > 0$ and $f_{xx}(2,-3) = 2 > 0$, so $(2,-3)$ gives the only relative minimum (and thus the absolute minimum) of f. The required points are $(2, -3, \pm 1)$, and the required distance is $d = \sqrt{f(2,-3)} = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$.