# ( Section 13.5 )

**3.** $\frac{dw}{dt} = \frac{\partial w}{\partial r}\frac{dr}{dt} + \frac{\partial w}{\partial s}\frac{ds}{dt} = (\cos s + s\cos r)\left(-2e^{-2t}\right) + (-r\sin s + \sin r)\left(3t^2 - 2\right)$

$= -2(\cos s + s\cos r)\,e^{-2t} + (\sin r - r\sin s)\left(3t^2 - 2\right)$

**10.** $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = (y\cos xy)\,3(u+v)^2 + (x\cos xy)(0) = 3(u+v)^2\,y\cos xy$ and

$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = (y\cos xy)\,3(u+v)^2 + (x\cos xy)\left(\frac{1}{2\sqrt{v}}\right)$

**22.** $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{v^2}{(u^2+v^2)^{3/2}}(1) - \frac{uv}{(u^2+v^2)^{3/2}}\cdot\cos\pi(y+z)$ and

$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z} = \frac{v^2}{(u^2+v^2)^{3/2}}(3) - \frac{uv}{(u^2+v^2)^{3/2}}\cdot(-\pi x)\sin\pi(y+z)$. If $x = 0$, $y = 1$, and $z = 1$, then

$u = 5$ and $v = 0$, so $\frac{\partial w}{\partial x} = \frac{0}{(25+0)^{3/2}} - \frac{5(0)}{125}(1) = 0$ and $\frac{\partial w}{\partial z} = 0 + \frac{5(0)}{125} = 0$.

**47.** $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}e^r\cos\theta + \frac{\partial u}{\partial y}e^r\sin\theta = e^r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$ and

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\left(-e^r\sin\theta\right) + \frac{\partial u}{\partial y}e^r\cos\theta = e^r\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta\right)$, so

$$e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right] = e^{-2r}\left[e^{2r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)^2 + e^{2r}\left(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta\right)^2\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(\frac{\partial u}{\partial y}\right)^2\left(\cos^2\theta + \sin^2\theta\right) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

**51.** Let $u = x + at$ and $v = x - at$. Then $z = f(x+at) + g(x-at) = f(u) + g(v)$,

so $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = f'(u)(a) + g'(v)(-a) = a\left[f'(u) - g'(v)\right] \Rightarrow$

$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial t}\left[f'(u) - g'(v)\right] = a\left[f''(u)(a) - g''(v)(-a)\right] = a^2\left[f''(u) + g''(v)\right]$

and $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = f'(u)(1) + g'(v)(1) = f'(u) + g'(v) \Rightarrow$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(u) + g'(v)\right] = f''(u)(1) + g''(v)(1) = f''(u) + g''(v)$. Therefore,

$\frac{\partial^2 z}{\partial t^2} = a^2\left[f''(u) + g''(v)\right] = a^2\frac{\partial^2 z}{\partial x^2}$.

## ( Section 13.6 )

**2.** Here $\mathbf{u} = \cos\frac{3\pi}{4}\,\mathbf{i} + \sin\frac{3\pi}{4}\,\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$, $D_{\mathbf{u}} f(x, y) = \dfrac{-x}{\sqrt{y^2 - x^2}}\left(-\dfrac{\sqrt{2}}{2}\right) + \dfrac{y}{\sqrt{y^2 - x^2}}\left(\dfrac{\sqrt{2}}{2}\right)$, and $D_{\mathbf{u}} f(4, 5) = -\frac{4}{3}\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{5}{3}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$.

**7.** $f_x(x, y) = \frac{\partial}{\partial x}(x \sin y + y \cos x) = \sin y - y \sin x$ and $f_y(x, y) = x \cos y + \cos x$, so $\nabla f\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \left[(\sin y - y \sin x)\,\mathbf{i} + (x \cos y + \cos x)\,\mathbf{j}\right]_{(\pi/4, \pi/2)} = \left[1 - \frac{\pi}{2}\left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \frac{4 - \sqrt{2}\pi}{4}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$.

**17.** Here $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{-2\mathbf{i} + 3\mathbf{j}}{\sqrt{(-2)^2 + 3^2}} = -\frac{2\sqrt{13}}{13}\mathbf{i} + \frac{3\sqrt{13}}{13}\mathbf{j}$, $f_x(x, y) = \sin^2 y$, and $f_y(x, y) = 2x \sin y \cos y = x \sin 2y$, so $D_{\mathbf{u}} f\left(-1, \frac{\pi}{4}\right) = f_x\left(-1, \frac{\pi}{4}\right)\left(-\frac{2\sqrt{13}}{13}\right) + f_y\left(-1, \frac{\pi}{4}\right)\left(\frac{3\sqrt{13}}{13}\right) = \frac{1}{2}\left(-\frac{2\sqrt{13}}{13}\right) + (-1)\left(\frac{3\sqrt{13}}{13}\right) = -\frac{4\sqrt{13}}{13}$.

**29.** Here $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \dfrac{\mathbf{i} + 3\mathbf{j}}{\sqrt{1^2 + 3^2}} = \frac{\sqrt{10}}{10}\mathbf{i} + \frac{3\sqrt{10}}{10}\mathbf{j}$, $f_x(x, y) = 3x^2$, and $f_y(x, y) = 3y^2$, so $D_{\mathbf{u}} f(1, 2) = f_x(1, 2)\left(\frac{\sqrt{10}}{10}\right) + f_y(1, 2)\left(\frac{3\sqrt{10}}{10}\right) = 3\left(\frac{\sqrt{10}}{10}\right) + 12\left(\frac{3\sqrt{10}}{10}\right) = \frac{39\sqrt{10}}{10}$.

**37.** $\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} = \dfrac{2}{1 + (2x + y)^2}\mathbf{i} + \dfrac{1}{1 + (2x + y)^2}\mathbf{j}$, so $\nabla f(0, 0) = 2\mathbf{i} + \mathbf{j}$ and a vector in the desired direction is $\mathbf{v} = -\nabla f(0, 0) = -2\mathbf{i} - \mathbf{j}$. The maximum rate of decrease of $f$ at $P$ is $|-\nabla f(0, 0)| = \sqrt{4 + 1} = \sqrt{5}$.