Homework-1

Exercises of Section(7.1 - 7.2)

2018.03.06

- **(Section 7.1)**

Evaluate the integral .

(12).

$$\int \sin^{-1} x \; dx$$

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- Sol :

$$\text{Let } u = \sin^{-1} x \;,\; du = \frac{1}{\sqrt{1-x^2}} dx \quad ; \quad dv = dx \;,\; v = x$$

$$\Rightarrow \int \sin^{-1} x \; dx = uv - \int v du = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Let $y = 1 - x^2$ then $dy = -2x \; dx$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int y^{-1/2} \; dy = -y^{1/2} + C = -\sqrt{1-x^2} + C$$

Hence

$$\int \sin^{-1} x \; dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

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(14).

$$\int \frac{\ln t}{\sqrt{t}} \; dx$$

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- Sol :

Let $u = \ln t$, $du = \frac{dt}{t}$; $dv = t^{-1/2} dt$, $v = 2t^{1/2}$ then

$$\int \frac{\ln t}{\sqrt{t}} \; dx = 2t^{1/2} \ln t - 2 \int t^{1/2} \cdot t^{-1} \; dt$$

$$= 2t^{1/2} \ln t - 2 \int t^{-1/2} \; dt = 2t^{1/2} \ln t - 4t^{1/2} + C = 2t^{1/2}(\ln t - 2) + C$$

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(16).

$$\int \boldsymbol{e^{-x} \sin x \; dx}$$

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- Sol :

Let $u = e^{-x}$, $du = -e^{-x}dx$; $dv = \sin x dx$, $v = -\cos x$

$$\Rightarrow \int e^{-x} \sin x \; dx = -e^{-x} \cdot \cos x - \int e^{-x} \cos x \; dx$$

Do integration by parts again

Let $u = e^{-x}$, $du = -e^{-x}dx$; $dv = \cos x dx$, $v = \sin x$

$$\Rightarrow \int e^{-x} \sin x \; dx = -e^{-x} \cdot \cos x - \left(e^{-x} \cdot \sin x + \int e^{-x} \sin x \; dx \right)$$

$$\Rightarrow 2 \int e^{-x} \sin x \; dx = -e^{-x} \cdot \cos x - e^{-x} \cdot \sin x + C$$

Hence

$$\int e^{-x} \sin x \; dx = \frac{1}{2}\Big(- e^{-x} \cdot \cos x - e^{-x} \cdot \sin x \Big) + C = -\frac{1}{2}e^{-x}\Big(\cos x + \sin x \Big) + C$$

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(35).

$$\int_0^{1/2} \cos^{-1} x \; dx$$

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- Sol :

First find $\int \cos^{-1} x \; dx =??$

Let $u = \cos^{-1} x$, $du = \dfrac{-1}{\sqrt{1-x^2}} \; dx$; $dv = dx$, $v = x$

$$\int \cos^{-1} x \; dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \; dx$$

Let $y = 1 - x^2$ then $dy = -2x \; dx$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int y^{-1/2} \; dy = -y^{1/2} + C = -\sqrt{1-x^2} + C$$

Hence

$$\int \cos^{-1} x \ dx = x \cos^{-1} x - \sqrt{1-x^2} + C$$

Therefore

$$\int_0^{1/2} \cos^{-1} x \ dx = x \cos^{-1} x \Big|_0^{1/2} - \sqrt{1-x^2} \Big|_0^{1/2} = \frac{1}{2} \cdot \frac{\pi}{3} - \frac{\sqrt{3}}{2} + 1 = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$$

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(67). ***Suppose that*** f'' ***is continuous on*** $[1,3]$ ***and*** $f(1) = 2$ ***,*** $f(3) = -1$ ***,*** $f'(1) = 2$ ***and*** $f'(3) = 5$***. Evaluate***

$$\int_1^3 x f''(x) \ dx$$

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○ Sol :

Let $u = x$, $du = dx$; $dv = f''(x)dx$, $v = f'(x)$

then

$$\int x f''(x) \ dx = x f'(x) - \int f'(x) dx = x f'(x) - f(x) + C$$

Hence

$$\int_1^3 x f''(x) \ dx = x f'(x) \Big|_1^3 - f(x) \Big|_1^3 = 3 \cdot 5 - 2 - (-1 - 2) = 16$$

- **(Section 7.2)**

Evaluate the integral .

(10).

$$\int \sin^2 2x \cos^4 2x \ dx$$

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- Sol :

$$\int \sin^2 2x \cos^4 2x \ dx = \int \frac{1-\cos(4x)}{2} \cdot \Big(\frac{1+\cos(4x)}{2}\Big)^2 \ dx$$
$$= \frac{1}{8}\int (1-\cos(4x))(1+\cos(4x))^2 \ dx$$
$$= \frac{1}{8}\int (1-\cos(4x)) \cdot (1+2\cos(4x)+\cos^2(4x)) \ dx$$
$$= \frac{1}{8}\int 1+\cos(4x)-\cos^2(4x)-\cos^3(4x) \ dx$$
$$= \frac{1}{8}\int \sin^2(4x)+\cos(4x)\Bigg(1-\cos^2(4x)\Bigg) \ dx$$
$$= \frac{1}{8}\int \left(\frac{1-\cos(8x)}{2}\right) + \cos(4x)\sin^2(4x) \ dx$$
$$= \frac{1}{16}\int (1-\cos(8x))dx + \frac{1}{32}\int u^2 du \quad ; \quad (\ u = \sin(4x) \ , \ du = 4\cos(4x)dx \ )$$
$$= \frac{1}{16}(x - \frac{1}{8}\sin(8x)) + \frac{1}{32}\cdot\frac{1}{3}u^3 + C$$
$$= \frac{1}{16}\left(x - \frac{1}{8}\sin(8x) + \frac{1}{6}\sin^3(4x)\right) + C$$

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(13).

$$\int_0^{\pi} \sin^2 x \cos^4 x \; dx$$

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○ Sol :

$$\begin{aligned}
\int \sin^2 x \cos^4 x \; dx &= \int \frac{1-\cos 2x}{2} \cdot \left(\frac{1+\cos 2x}{2}\right)^2 \; dx \\
&= \frac{1}{8}\int (1-\cos(2x))(1+\cos(2x))^2 \; dx \\
&= \frac{1}{8}\int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x)) \; dx \\
&= \frac{1}{8}\int 1+\cos(2x)-\cos^2(2x)-\cos^3(2x) \; dx \\
&= \frac{1}{8}\int \sin^2(2x)+\cos(2x)(1-\cos^2(2x))dx \\
&= \frac{1}{8}\int \frac{1-\cos(4x)}{2} \; dx + \frac{1}{8}\int \cos(2x)\sin^2(2x)dx \\
&= \frac{1}{16}(x-\frac{1}{4}\sin(4x)) + \frac{1}{8}\cdot\frac{1}{2}\cdot\frac{1}{3}\sin^3(2x)+C \\
&= \frac{1}{16}\left(x-\frac{1}{4}\sin(4x)+\frac{1}{3}\sin^3(2x)\right)+C
\end{aligned}$$

Hence

$$\int_0^{\pi} \sin^2 x \cos^4 x \; dx = \frac{1}{16}\left(x-\frac{1}{4}\sin(4x)+\frac{1}{3}\sin^3(2x)\right)\Bigg|_0^{\pi} = \frac{\pi}{16}$$

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(24).

$$\int_0^{\pi/4} \sec^2 x \tan^2 x \; dx$$

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○ Sol :

Let $u = \tan x$ then $du = \sec^2 x dx$

$$\int \sec^2 x \tan^2 x \; dx = \int u^2 \; du == \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

Hence

$$\int_0^{\pi/4} \sec^2 x \tan^2 x \; dx = \frac{1}{3}\tan^3 x\Bigg|_0^{\pi/4} = \frac{1}{3}$$

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(42).

$$\int \frac{\sin^3 x}{\sec^2 x} dx$$

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∘ Sol :

$$\begin{aligned} \int \frac{\sin^3 x}{\sec^2 x} dx &= \int \cos^2 x \sin^2 x \sin x \ dx = \int \cos^2 x (1 - \cos^2 x) \sin x \ dx \\ &= \int (\cos^2 x - \cos^4 x) \sin \ dx \end{aligned}$$

Let $u = \cos x$, $du = -\sin x dx$ then

$$\int \frac{\sin^3 x}{\sec^2 x} dx = -\int u^2 - u^4 du = -\frac{1}{3} u^3 + \frac{1}{5} u^5 + C = -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C$$

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(44).

$$\int_0^{\pi/2} \frac{\sin t}{1 + \cos t} dt$$

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∘ Sol :

Let $u = 1 + \cos t$, $du = -\sin t dt$

$$\int \frac{\sin t}{1 + \cos t} dt = -\int u^{-1} du = -\ln |u| + C = -\ln |1 + \cos t| + C$$

Hence

$$\int_0^{\pi/2} \frac{\sin t}{1 + \cos t} dt = -\ln |1 + \cos t| \Big|_0^{\pi/2} = \ln 2$$
