Calculus Exam-1 (107.04.10)

- **1. (15%) Evaluate**

$$\boldsymbol{\int \frac{\sqrt{x^2-16}}{x}\, dx}$$

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○ Sol :

Let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta d\theta$

$$\int \frac{\sqrt{x^2-16}}{x}\, dx = \int \frac{\sqrt{16\sec^2\theta - 16}}{4\sec\theta}\, 4\sec\theta\tan\theta d\theta = \int 4\tan^2\theta\ d\theta$$
$$= 4\int (\sec^2\theta - 1)d\theta = 4\int \sec^2\theta\ d\theta - 4\theta = 4\tan\theta - 4\theta + C$$

Since $\sec\theta = \frac{x}{4}$ then $\theta = \sec^{-1}\left(\frac{x}{4}\right)$ and $\tan\theta = \frac{\sqrt{x^2-16}}{4}$

$$\int \frac{\sqrt{x^2-16}}{x}\, dx = 4\left(\frac{\sqrt{x^2-16}}{4}\right) - 4\sec^{-1}\left(\frac{x}{4}\right) + C = \frac{\sqrt{x^2-16}}{4} - 4\sec^{-1}\left(\frac{x}{4}\right) + C$$

- **2. (15%) Evaluate**

$$\boldsymbol{\int_2^4 \frac{3x-5}{(x-1)^2}\, dx}$$

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○ Sol :

$$\frac{3x-5}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \ \Rightarrow\ \frac{3x-5}{(x-1)^2} = \frac{Ax-A+B}{(x-1)^2} \ \Rightarrow\ A = 3 \quad ; \quad B = -2$$

Hence

$$\int_2^4 \frac{3x-5}{(x-1)^2}\, dx = \int \frac{3}{x-1} - \frac{2}{(x-1)^2}\, dx = 3\ln|x-1| + 2(x-1)^{-1} + C$$

Hence

$$\int_2^4 \frac{3x-5}{(x-1)^2}\, dx = 3\ln|x-1|\Big|_2^4 + 2(x-1)^{-1}\Big|_2^4 = 3\ln 3 + 2\cdot\frac{1}{3} - 2 = 3\ln 3 - \frac{4}{3}$$

- **3. (15%) Show that the series**

$$\sum_{n=1}^{\infty} \frac{4}{4n^2-1}$$

is convergent, and find its sum.

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○ Sol :

Rewrite the general term

$$a_n = \frac{4}{4n^2-1} = \frac{4}{(2n-1)(2n+1)} = \frac{2}{2n-1} - \frac{2}{2n+1}$$

Then we write the nth partial sum of the series as

$$\begin{aligned} S_n = \sum_{k=1}^{n} \frac{4}{4n^2-1} &= \sum_{k=1}^{n} \left(\frac{2}{2k-1} - \frac{2}{2k+1} \right) \\ &= \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) + \cdots + \left(\frac{2}{2n-1} - \frac{2}{2n+1}\right) \\ &= 2 - \frac{2}{2n+1} \end{aligned}$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2 - \frac{2}{2n+1}\right) = 2$$

Hence the given series is convergent and its sum is 2.

- **4. (15%) Use the integral test to determine whether**

$$\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n^2+1}}$$

converges or diverges.

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○ Sol :

$$\text{Consider} \quad \sum_{n=1}^{\infty} \frac{1}{n^2+1} = \sum_{n=1}^{\infty} a_n \quad ; \quad \text{Let} \quad f(x) = \frac{1}{x^2+1}$$

$f(x)$ is a continuous, positive and decreasing function on $[1, \infty)$, hence we can use the integral test on the series for $a_n = f(n)$.

$$\int_1^{\infty} f(x)dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\, dx = \lim_{b\to\infty} \tan^{-1} x \Big|_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \;\Rightarrow\; \textit{converges}$$

Since $\int_1^{\infty} f(x)dx$ converges then the series is convergent.

- **5. (10%) Evaluate**

$$\int_1^e \ln x \, dx$$

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○ Sol :

Let $\ln x = u$, $dx = dv \Rightarrow du = \frac{1}{x}\, dx$, $v = x$

then

$$\int \ln x \, dx = x \cdot \ln x - \int x \cdot \frac{1}{x}\, dx = x \ln x - x + C$$

Hence

$$\int_1^e \ln x \, dx = x \ln x \Big|_1^e - x \Big|_1^e = e - (e-1) = 1$$

- **6. (10%) Evaluate**

$$\int_2^4 \frac{1}{\sqrt{4-x}}\, dx$$

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○ Sol :

Let $u = 4 - x$, $du = -dx \Rightarrow \int \frac{1}{\sqrt{4-x}}\, dx = \int -u^{-1/2}\, du = -2u^{1/2} + C = -2\sqrt{4-x} + C$

Note that the integrand has an infinite discontinuity at $x = 4$.

Thus

$$\int_2^4 \frac{1}{\sqrt{4-x}}\, dx = \lim_{c \to 4^-} \int_2^c \frac{1}{\sqrt{4-x}}\, dx = \lim_{c \to 4^-} \left(-2\sqrt{4-x}\right)\Big|_2^c = \lim_{c \to 4^-} \left(-2\sqrt{4-c} + 2\sqrt{2}\right) = 2\sqrt{2}$$

- **7. (10%) Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ is divergent**

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○ Sol : Using the Limit Comparison :

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2$, it is divergent.

Thus we can conclude that the given series is divergent as well.

- **8. (10%) Determine whether the series**

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3n}{4n^2-1}$$

converges or diverges.

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○ Sol :

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad ; \quad a_n = \frac{3n}{4n^2-1}$$

(i) Show that $a_n \geq a_{n+1}$ for all n :

$$\text{Let} \quad f(x) = \frac{3x}{4x^2-1} \;\Rightarrow\; f'(x) = \frac{3(4x^2-1) - 3x(8x)}{(4x^2-1)^2} = \frac{-12x^2-3}{(4x^2-1)^2}$$

$$\Rightarrow\; f'(x) < 0 \text{ for } x \geq 0$$

it means that $a_n \geq a_{n+1}$ for all n.

(ii) Show that $\lim_{n\to\infty} a_n = 0$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{4n^2-1} = \lim_{n\to\infty} \frac{\frac{3}{n}}{4-\frac{1}{n^2}} = 0$$

Since (i) and (ii) conditions are both satisfied, we conclude that the series is convergent.