1

2

3

4 Integration

4.1 Antiderivatives and indefinite integral

Definition: A function F is "an" antiderivative of f on the inteval I if $F^{'}(x) = f(x)$ for all $x \in I$.

Theorem 4.1 If F is an antiderivative of f, then G is an antiderivative of f on the inteval I iff $G(x) = F(x) + C$, where C is a const.

proof. Clearly, $F(x) + C$ is an anti-derivative of f.
($\Leftarrow$) If $G(x)$ is an antiderivative of f. Let

$$H(x) = G(x) - F(x).$$

If H is NOT const. $\exists\ a < b$ in I s.t. $H(a) \neq H(b)$ and by MVT, $\exists\ a < c < b$ s.t.

$$H^{'}(c) = \frac{H(b) - H(a)}{b - a} \neq 0.$$

Now since G, F are anti-derivative of f, $G^{'}(x) = F^{'}(x) = f(x)$.
In particular $G^{'}(x) = F^{'}(x)$, hence $H^{'}(c) = 0$.
This contradiction implies that $H(x) = const. = c$ and

$$G(x) = F(x) + C.$$

□

We use the notation

$$\int f(x)dx$$

for anti-derivative of f. It is also called **indefinite integral**.

Basic integration rules:

$$1. \int f'(x)dx = F(x) + C.$$
$$2. \frac{d}{dx}\int f(x)dx = f(x).$$

p.250.
Example 4.5.6.

4.2 Area

Upper and lower sum
f: continuous. Approximation of the area
1. Subdivides the inteval $[a, b]$ into n subintervals
with width $\triangle x = \frac{b-a}{n}$.
2. In each subinterval
$f(m_i)$ = min. value of $f(x)$ in the i^{th} subinterval
$f(M_i)$ = max. value of $f(x)$ in the i^{th} subinterval
(can be done by extreme value thm for continuous function
$f(x)$ on closed intervals.)

The lower sum is defined to be
lower sum= $s(n) = \sum_{i=1}^{n} f(m_i)\triangle x$
upper sum= $S(n) = \sum_{i=1}^{n} f(M_i)\triangle x$
$s(n) \leq$ Area of the region $\leq S(n)$

Theorem 4.3 f continuous nonnegative on $[a, b]$.
Then $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n)$.
Note that for any x in the i^{th} inteval $f(m_i) \leq f(x) \leq f(M_i)$, and since both limit in thm 4.3 are equal, by squeeze thm, we can make the following

Definition: f: continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis and the two lines $x = a$, $x = b$ is
. Area $= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x_i$, $x_{i-1} \leq c_i \leq x_i$,

where $\triangle x = \frac{b-a}{n}$.

Ex6 p266.

4.3 Riemann sums and definite integrals

Riemann sums Definition: f: defined on a closed interval $[a, b]$.
$\triangle$: partition of $[a, b]$ given by
$a = x_0 < x_1 < x_2 < \cdots < x_n = b$
$\triangle x_i$: the width of the i^{th} interval. subunterval.
c_i: any point in the i^{th} subinterval.
Then the sum

. $$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x_i,\ x_{i-1} \le c_i \le x_i,$$

is called the **Riemann sums** of f for the partition $\triangle$.
$\|\triangle\|$: width of the largest subinterval in the partition $\triangle$.
Note $\|\triangle\| \to 0$ as $n \to \infty$.

Definition: If f is defined on $[a, b]$ and the limit

. $$\lim_{\|\triangle\|\to 0} \sum f(c_i)\triangle x_i,\ x_{i-1} \le c_i \le x_i,$$

exists, the f is integrable on $[a, b]$ and the limit is denoted by

. $$\lim_{\|\triangle\|\to 0} \sum f(c_i)\triangle x_i = \int_a^b f(x)dx,$$

and it is called the **definite integral** of f from a to b.
a: lower limit.
b: upper limit.

Theorem 4.4 If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Theorem 4.5 f: continuous, nonnegative on $[a, b]$, then

$$Area = \int_a^b f(x)dx.$$

Properties of definite integrals
Definition: 1. If f is defined at $x = a$, then we set $\int_a^a f(x)dx = 0$.
2. If f is integrable on $[a, b]$, then we set $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Theorem 4.6 If f is integrable on the three closed integrals determinied by a,b,c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem 4.7 If f,g are integrable on $[a,b]$, k any const., then

1. $$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$
2. $$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

Theorem 4.8 1. If f is integrable and nonnegative on $[a,b]$, then

$$0 \leq \int_a^b f(x)dx.$$

2. If f,g are integrable and $f(x) \geq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

Properties of definite integrals

1. If f is defined at $x = a$, then we set $\int_a^a f(x)dx = 0$.
2. If f is integrable on $[a,b]$, then we set $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
3. If f is integrable on $[a,b]$ and $c \in [a,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

4. If f and g are integrable on $[a,b]$ and k is a const., then

   1. $$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$
   2. $$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

5. If f is integrable and nonnegative on $[a,b]$, then

$$\int_a^b f(x)dx \geq 0.$$

6. If f and g are integrable on $[a,b]$ with $f(x) \geq g(x)$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

4.4 Fundamental Theorem of Calculas

Theorem 4.9 The fundamental theorem of calculas.
If f is a function continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

proof. Let $\triangle$ be the partition of $[a, b]$,

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Now $F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0)$

$$= \sum_{i=1}^{n} F(x_i) - F(x_{i-1}).$$

By MVT $\exists$ $c_i \in [x_{i-1}, x_i]$ s.t. $F(c_i)' = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$. Thus,

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\triangle x_i.$$

Now this is true for any function, by taking limit, we have

$$\int_a^b f(x)dx = F(b) - F(a).$$

$\square$

Examples. (i) $\int_1^4 3\sqrt{x}dx$, (ii) $\int_0^2 |2x - 1|dx$ $\quad x = \frac{1}{2}$ to divide.

Remark: Notation we write

$$\int_a^b f(x)dx = F(x) \mid_a^b = F(b) - F(a).$$

Note that it is NOT necessary to include a const. C in the antiderivative $F(x)$.
Theorem 4.10 MVT for integrals.
If f is a continuous on $[a, b]$, then $\exists$ a $c \in [a, b]$ s.t.

$$\int_a^b f(x)dx = f(c)(b - a).$$

proof. 1. If f =const. and any pt. c in $[a,b]$ will do the work.
2. If $f \neq$const. By extreme value thm $\exists\, m, M \in [a,b]$ s.t.
$f(M)$ is the maximum and $f(m)$ is the minimum, i.e., we have

$$f(m) \leq f(x) \leq f(M)\ \forall x \in [a,b].$$

$\Rightarrow f(m)(b-a) = \int_a^b f(m)dx \leq \int_a^b f(x)dx \leq \int_a^b f(M)dx = f(M)(b-a)$

$$f(m) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(M).$$

Now use the Intermediate Value thm, $\exists$ a $c \in [a,b]$ s.t.

$$\int_a^b f(x)dx = f(c)(b-a).$$

□

The value $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ is defined to be the average of f over $[a,b]$.

Theorem 4.11 2^{nd} fundamental theorem of calculas.
If f is a continuous on an open interval I containing a, then for any $x \in I$,
. $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

proof. Let $F(x) = \int_a^x f(t)dt$, then

$$F(x)' =_{\triangle x \to 0} \frac{F(x+\triangle x)-F(x)}{\triangle x}.$$

Now $F(x+\triangle x)-F(x) = \int_a^{x+\triangle x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\triangle x} f(t)dt = f(c)(x+\triangle x - x) = f(c)\triangle x$, $c \in [x, x+\triangle x]$. Here we use the MVT for integral. So as $\triangle x \to 0$ $c \to x$,
. $F(x)' =_{\triangle x \to 0} \frac{1}{\triangle x}(f(c)\triangle x) = f(x)$.

□

Example. $F(x) = \int_{\frac{\pi}{2}}^{x^3} costdt$.
Solve: Let $u = x^3$ then $F(x)' = \frac{dF}{du}\frac{du}{dx} = \frac{d}{du}(\int_{\frac{\pi}{2}}^{u} costdt)\frac{du}{dx} = cosu \cdot (3x^2) = 3x^2cosx^3$.

4.5 Integration by substitution

Recall the **chain rule**: If $y = F(u)$ and $u = g(x)$ are differentiable, then

$$\frac{d}{dx}F(g(x)) = F(g(x))^{'}g(x).$$

Now if F is an anti-derivative of the function f, we have the following **change of variable theorem**.

Theorem 4.12 g: differentiable function with range in an interval I and f is a function continuous on I. If F is an anti-derivative of the function f, then

$$\int f(g(x))g^{'}(x)dx = F(g(x)) + C.$$

In real computation, we do the following:

$$\textit{Let } u = g(x), \textit{ then } du = g(x)^{'}dx \textit{ and}$$
$$\int f(g(x))g^{'}(x)dx = \int f(u)du = F(u) + C.$$

Examples.

1. $\int 2x(x^2+1)^4dx$
2. $\int x^2\sqrt{x^3+1}dx$
3. $\int 2sec^2x(tanx+3)dx$
4. $\int x\sqrt{2x-1}dx \qquad u = 2x-1 \quad x = \frac{u+1}{2} \quad dx = \frac{1}{2}du.$
5. $\int sin^2 3xcos3xdx \qquad \int (g(x))^n g(x)^{'}dx = \frac{g(x)^{n+1}}{n+1} + C.$
6. $\int (2x+1)(x^2+x)dx$
7. $\int \frac{-4x}{(1-2x^2)^2}dx$

Change of variable for definite integrals

Theorem 4.14 If $u = g(x)$ has a continuous derivative on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g^{'}(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Examples. $\int_1^5 \frac{x}{\sqrt{2x-1}}dx$ with $u = \sqrt{2x-1}$ $\quad x = \frac{u^2+1}{2}$

Integration of even and odd functions
f is even if $f(-x) = f(x)$
f is odd if $f(-x) = -f(x)$.
Then we have
Theorem 4.15 f: integrable on $[-a, a]$.
1. If f is even, then $\int_{-a}^a f(x)dx = 2\int_{-a}^a f(x)dx$.
2. If f is odd, then $\int_{-a}^a f(x)dx = 0$.

5

5.1 Natural logarithmic function

General power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

For $n = -1$, we define the natural logarithmic function to be

$$lnx = \int_1^x \frac{dt}{t}, x > 0$$

The domain of the natural logarithmic function is the set of all positive real numbers. By 2^{nd} fund. thm of cal., lnx is an anti-derivative of the function $\frac{1}{x}$.

Theorem 5.1 The natural logarithmic function has the following properties.
1. The domain is $(0, \infty)$ and the image is $(-\infty, \infty)$.
2. The function is continuous, increasing and 1-1.
3. The graph is concave downward.

proof. 2. $\frac{d}{dx}lnx = \frac{1}{x} > 0$ for $x > 0$.
f: diff $\Rightarrow$ f: continuous, increasing and 1-1.
3. $\frac{d^2}{d^2x}lnx = -\frac{1}{x^2} < 0$ for $x > 0$, hence concave downward. □

Now $\lim_{x\to 0^+} lnx = -\infty$, $\lim_{x\to\infty} lnx = \infty$.

Theorem 5.2 If $a, b > 0$, n: rational, then we have
1. $ln1 = 0$.
2. $ln(ab) = lna + lnb$.
3. $ln(a^n) = nlna$.
4. $ln(\frac{a}{b}) = lna - lnb$.

proof. 1. by definition, $ln1 = \int_1^1 \frac{t}{dt} = 0$.
2. $\frac{d}{dx}ln(ax) = \frac{a}{ax} = \frac{1}{x}$, $\frac{d}{dx}(lna + lnx) = \frac{1}{x}$
$\Rightarrow ln(ax) = lna + lnx + C$.
Let $x = 1$, we have $lna = lna + ln1 + C$,

$$\Rightarrow C = 0.$$

So $ln(ab) = lna + lnb$.
3. Comparing the derivatives of lnx^n and $nlnx$.
4. Special case of 2. □

Theorem 5.3 If u is a differentiable function of x, $u > 0$, then by chain rule

$$\frac{d}{dx}lnu = \frac{1}{u}\frac{d}{dx}u = \frac{u'}{u}$$

Theorem 5.4 If u is a differentiable function of x, $u \neq 0$, then

$$\frac{d}{dx}ln|u| = \frac{u'}{u}$$

Definition: Denote the letter e to be the positive number s.t.

$$lne = \int_1^e \frac{d}{dt} = 1.$$

$$e \approx 2.718281829459045\cdots$$

Examples 1. differentaible $f(x) = ln\frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$
2. Find the derivative of $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$, $x \neq 2$. taking ln.
3. Find the derivative of $f(x) = ln|cosx|$.

5.2 Integration of natural logarithmic function

Theorem 5.5

$$\int \frac{1}{x} dx = ln|x| + C, \qquad \int \frac{u'}{u} = ln|u| + C.$$

Example 1. 1. $\int \frac{2}{x} dx = 2ln|x| + C$
2. $\int \frac{dx}{4x-1} = \frac{1}{4} ln|4x-1| + C$
3. Find the area of the region bdd by $y = \frac{x}{x^2+1}$ and the lines $x = 0$, $x = 3$.
4. $\int \frac{3x^2+1}{x^3+x} dx = ln|x^3 + x| + C$
. $\int \frac{sec^2 x}{tanx} dx = ln|tanx| + C$
5. $\int \frac{x^2+x+1}{x^2+1} dx$
6. $\int \frac{2x}{(x+1)^2} dx$ u=x+1.
7. $\int \frac{1}{xlnx} dx$
8. Find $\int tanx dx = -ln|cosx| + C$
9. Find $\int secx dx = \int secx \frac{secx+tanx}{secx+tanx} dx$ $u = secx + tanx$.
10. $\int_0^{\frac{\pi}{4}} \sqrt{1 + tan^2 x} dx = \int_0^{\frac{\pi}{4}} \sqrt{sec^2 x} dx = \int_0^{\frac{\pi}{4}} secx dx$

5.3 Inverse functions

Definition: A finction g is the **inverse function** of the function f if
$f(g(x)) = x$ for each x in the domain of g and
$g(f(x)) = x$ for each x in the domain of f.
In notation we write g by f^{-1}.
If we write $f(x) = y$, then $f^{-1}(f(x)) = f^{-1}(y) = x$.

Theorem 5.6 The graph of f contains the point (a, b) iff the graph of f^{-1} contains the point (b, a).

proof. If (a, b) is in the graph of f, then $f(a) = b$. So $f^{-1}(b) = f^{-1}(f(a)) = a$. □

The proof implies that the graph of f and the graph of f^{-1} are symmetric w.r.t. the line $y = x$.

Theorem 5.7 (Existence of inverse) 1. A function f has an inverse iff f is one-to-one.

2. If f is strictly monotonic on its domain, then it is one-to-one and hence it has an inverse.

proof. 2. f is one-to-one means $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, i.e., $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. If $x_1 \neq x_2$ and f is strictly monotonic, then either $x_1 < x_2$ or $x_1 > x_2$. In either case $f(x_1) \neq f(x_2)$, so f is one-to-one and by 1, f has an inverse. □

Remark: $y = f(x)$. If we can solve $x = g(y)$ then $y = f(g(y))$ hence g is an inverse of f.

Examples 1. a. $f(x) = x^3 + x - 1$ b. $f(x) = x^3 - x + 1$ $f(-1) = f(1) = f(0) = 1$. $y = 1$ horizontal line test.
2. $f(x) = \sqrt{2x - 3}$
3. $f(x) = sinx$.

Derivative of an inverse function
Theorem 5.8 Let f be a function with domain I. If f has an inverse, then the following are true.
1. If f is continuous on I, then f^{-1} is continuous on its domain.
2. If f is increasing(decreasing) on I, then f^{-1} is increasing(decreasing) on its domain.
3. If f is differentiable at c with $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Theorem 5.9 If f is differentiable on an interval I and f has an inverse g, then g is differentiable at any $y = f(x)$, where $f'(x) \neq 0$. In fact, we have at such point,

$$\frac{d}{dx} f^{-1}(y) = \frac{1}{f'(x)},$$

where $y = f(x)$.
(chain rule)

Example 2 $f(x) = \frac{1}{4}x^3 + x - 1$.
a. what is the value of $f^{-1}(y)$ when $y = f(x) = 3$.
b. what is the value of $(f^{-1})'(y)$ when $y = f(x) = 3$.

5.4 Exponential functions

The function $f(x) = lnx$ is increasing(why?) on its entire domain, hence it has an inverse function f^{-1}. the domain of f^{-1} is the set of all real numbers, and its range is $x > 0$.

If x is rational, then

$$ln(e^x) = xln(e) = x \cdot 1 = x.$$

Definition: The inverse function of the natural logarithmic function $f(x) = lnx$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x,$$

i.e., $y = e^x$ iff $lny = x$.
$ln(e^x) = x$. $e^{lnx} = x$.
Theorem 5.10 operations
$a, b \in \mathbb{R}$, then (i) $e^a e^b = e^{a+b}$, (ii) $\frac{e^a}{e^b} = e^{a-b}$.

proof. $ln(e^{a+b}) = ln(e^a) + ln(e^b) = a + b = ln(e^a \cdot e^b)$. □

Properties:
1. The domain of e^x is $(-\infty, \infty)$ and the image is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing and 1-1 on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \to -\infty} e^x = 0$, $\lim_{x \to \infty} e^x = \infty$.

Theorem 5.11 u: differentiable function of x.
1. $\frac{d}{dx} e^x = e^x$
2. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$.

proof. $ln(e^x) = x$
$\frac{d}{dx} ln(e^x) = 1$
$\frac{1}{e^x} \frac{d}{dx} e^x = 1$. □

Example $f(x) = xe^x$. $f'(x) = ?$

Theorem 5.11 u: differentiable function of x. Then
(i) $\int e^x dx = e^x + C$ (ii) $\int e^u u'(x) dx = e^{u(x)} + C$.

Example 1. $\int e^{3x+1}dx$ 2. $\int 5xe^{-x^2}dx$ 3. $\int \frac{e^{\frac{1}{x}}}{x^2} =?$
4. $\int_0^1 \frac{e^x}{1+e^x}dx$.

5.5 Bases other than e

Definition: If a is a positive real number($a \neq 1$), x is any real number, then we define a^x to be

$$a^x = e^{(lna)x}$$

exponentail with base a. If $a = 1$, then $y = 1^x = 1$ is a const. function.

Properties: (i) $a^0 = 1$, (ii) $a^x a^y = a^{x+y}$, (iii) $(a^x)^y = a^{xy}$.
Definition: If a ia a positive real number($a \neq 1$) and x is any real number, then the logarithmic function with base a is defined as

$$log_a x = \frac{lnx}{lna}.$$

Properties: (i) $log_a 1 = 0$, (ii) $log_a xy = log_a x + log_a y$, (iii) $log_a x^u = u log_a x$.

Examples (i) Solve $3^x = \frac{1}{81}$ and (ii) $log_2 x = -4$.

Theorem 5.13 $a > 0$, u: differentiable function of x.
1. $\frac{d}{dx}a^x = (lna)a^x$, 2. $\frac{d}{dx}(a^u) = (lna)a^u \frac{du}{dx}$.
3. $\frac{d}{dx}log_a x = \frac{1}{(lna)x}$, 4. $\frac{d}{dx}log_a u = \frac{1}{(lna)u}\frac{du}{dx}$.
5. $\int a^x dx = \frac{1}{lna}a^x + C$.

Examples (i) $\frac{d}{dx}2^{3x}$ (ii) $log_{10}(cosx)$ (iii) $\int 2^x dx$.

Theorem 5.14 n: any real number, u: differentiable function of x.
1. $\frac{d}{dx}x^n = nx^{n-1}$, 2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$.

Examples (i) $\frac{d}{dx}(x^e)$ (ii) $\frac{d}{dx}x^x$.

Theorem 5.15 $\lim_{x\to\infty}(1+\frac{1}{x})^x = \lim_{x\to\infty}(\frac{1+x}{x})^x = e$.

proof. Let $y = \lim_{x\to\infty}(1+\frac{1}{x})^x$. $lny = ln \lim_{x\to\infty}(1+\frac{1}{x})^x y = \lim_{x\to\infty} ln(1+\frac{1}{x})^x$
$= \lim_{x\to\infty} \frac{ln(1+\frac{1}{x})}{\frac{1}{x}}$ since ln is continuous.
Let $t = \frac{1}{x}$. Then $y = \lim_{x\to 0^+} \frac{ln(1+t)}{t} = \lim_{x\to 0^+} \frac{ln(1+t)-ln1}{t} = \frac{d}{dx} lnx$ at $x = 1$
$= \frac{1}{x}$ at $x = 1$
$= 1$.
Hence $lny = 1$, that is, $y = e$. □

5.6 Inverse trigonometric functions, differentiation

All trigonometric functions do NOT have inverse functions, because they are all periodic. To get an inverse, we have to restate their domain so that they are 1-1.

Definition: Inverse trigonometric functions

Function	domain	range
$y = arcsinx$ iff $siny = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = arccosx$ iff $cosy = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = arctanx$ iff $tany = x$	$-\infty \le x \le \infty$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = arccotx$ iff $coty = x$	$-\infty \le x \le \infty$	$0 \le y \le \pi$
$y = arcsecx$ iff $secy = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi$, $y \ne \frac{\pi}{2}$ or $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$y = arccscx$ iff $cscy = x$	$\lvert x\rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$

Fig 5.29 p 372

Properties: 1. If $-1 \le x \le 1$, and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, yhen

$$sin(arcsinx) = x \;\; and \;\; arcsin(siny) = y.$$

2. If $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then

$$tan(arctanx) = x \;\; and \;\; arctan(tany) = y.$$

3. If $|x| \ge 1$ $0 \le y \le \frac{\pi}{2}$ or $\frac{\pi}{2} \le y \le \pi$, then

$$sec(arcsecx) = x \;\; and \;\; arcsec(secy) = y.$$

Only true in these intervals, for example, $arcsin(sin\pi) = 0$ not π.

Theorem 5.16 Derivatives of inverse trigonometric functions
u: differentiable function in x.
$\frac{d}{dx}arcsinu = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}arccosu = \frac{-u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}arctanu = \frac{u'}{1+u^2}$ $\frac{d}{dx}arccotu = \frac{-u'}{1+u^2}$
$\frac{d}{dx}arcsecu = \frac{u'}{|u|\sqrt{u^2-1}}$ $\frac{d}{dx}arccscu = \frac{-u'}{|u|\sqrt{u^2-1}}$

proof. $\frac{d}{dx}sin(arcsinu) = \frac{d}{dx}u$
$\Rightarrow cos(arcsinu)\frac{d}{dx}arcsinu = \frac{u'}{\sqrt{1-u^2}}$
$\Rightarrow \frac{d}{dx}arcsinu = \frac{u'}{\sqrt{1-u^2}}$. □

Examples $y = arcsinx + x\sqrt{1-x^2}$, find $\frac{dy}{dx}$ =? $(2\sqrt{1-x^2})$

5.7 Inverse trigonometric functions, integration

From the differentiable formulas of inverse trigonometric functions, we have

Theorem 5.17 u: differentiable function of x, $a > 0$.
1. $\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\frac{u}{a} + C$
2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}arctan\frac{u}{a} + C$
3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}arcsec\frac{|u|}{a} + C$.

Examples 1. (i) $\int \frac{dx}{\sqrt{4-x^2}} = arcsin\frac{x}{2} + C$, (ii) $\int \frac{du}{2+9x^2} = \frac{1}{3\sqrt{2}}arctan\frac{3x}{\sqrt{2}} + C$, (iii) $\int \frac{dx}{x\sqrt{4x^2-9}} = \frac{1}{3}arcsec\frac{|2x|}{3} + C$.
2. $\int \frac{dx}{\sqrt{e^{2x}-1}} = arcsec\ e^x + C$. $u = e^x$ $du = udx$
3. $\int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}}dx + \int \frac{2}{\sqrt{4-x^2}}dx = -\sqrt{4-x^2} + 2aresin\frac{x}{2} + C$.
4. $\int \frac{dx}{x^2-4x+7} = \int \frac{dx}{(x-2)^2+3} = \frac{1}{\sqrt{3}}arctan\frac{x-2}{\sqrt{3}} + C$
5. Note the following similar but different integrals.
$\int \frac{dx}{x\sqrt{x^2-1}}$, $\int \frac{xdx}{\sqrt{x^2-1}}$, $\int \frac{dx}{\sqrt{x^2-1}}$
$\int \frac{dx}{xlnx}$, $\int \frac{lnx}{x}dx$, $\int lnxdx$.

6

7 Application of Integration

7.1 Area of a region between two curves

Find the area of the region between the graph of f and g.
Then

$$A = \int_a^b f(x)dx - \int_a^b f(x)dx$$
$$= \int_a^b (f(x) - g(x))dx.$$

$\Rightarrow$ If f and g are continuous on $[a, b]$ with $g(x) \leq f(x)$ for all $x \in [a, b]$, then the area of the region bounded by the graph of f and g and the lines $x = a$, $x = b$ is

$$A = \int_a^b (f(x) - g(x))dx.$$

Examples 1. Find the area of the region bounded by $f(x) = 2 - x^2$ and $g(x) = x$.
sol. $2 - x^2 = 2 \Rightarrow x^2 + x - 2 = 0$, $x = 1$, or -2.

$$A = \int_{-2}^{1} (2 - x^2 - x)dx = \frac{9}{2}.$$

2. Find the area of the region bounded by $sinx$ and $cosx$.
sol. $sinx = cosx \Rightarrow tanx = 1$, $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (sinx - cosx)dx = -cosx - sinx|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = 2\sqrt{2}.$$

3. Find the area of the region bounded by $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.
sol. $3x^3 - x^2 - 10x = -x^2 + 2x \Rightarrow 3x^3 - 12x = 0 \Rightarrow 3x(x-2)(x+2) = 0$
$\Rightarrow x = -2, 0, 2$.

$$A = \int_{-2}^{0} (f(x) - g(x))dx + \int_{0}^{2} (f(x) - g(x))dx = 24.$$

7.2 Volume: disk method

The disk method.

$$V = \int_a^b (R(x))^2 dx. \qquad \textit{horizontal revolution.}$$

$$V = \int_a^b ((R(x))^2 - (r(x))^2) dx.$$

Examples 2 p.458, 3 p.459, 4 p.
460

If we know the cross-section of a solid, then we may find the volume of the solid by
1. $A(x)$: the area of the cross section perpendicular to the x-axis.

$$V = \int_a^b A(x) dx.$$

2. $A(y)$: the area of the cross section perpendicular to the y-axis.

$$V = \int_c^d A(y) dy.$$

7.3 Shell method

Another way to find the volume of a solid is the shell method. Consider the following solid of revolution. The approximated shell has volume

$$\triangle V = 2\pi p(y) k(y) \triangle y.$$

The volume of the solid is then

$$V = 2\pi \int_c^d p(y) k(y) dy.$$

This is the shell method.

Examples p.469.
1. Find the volume of the solid formed by revolving the region bounded by

$x = e^{-y^2}$, and the y-axis about the x-axis.
sol.

$$0 \leq y \leq 1$$
$$V = 2\pi \int_0^1 ye^{-y^2} dy = \pi(1 - \frac{1}{e}).$$

2. Find the volume of the solid formed by revolving the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$ about the y-axis.

disk method

$$\begin{aligned} V &= \pi \int_0^1 (1^2 - 0^2) dy + \pi \int_1^2 (1^2 - (\sqrt{y-1})^2) dy \\ &= \pi \int_0^1 dy + \pi \int_1^2 (2 - y) dy = \frac{3\pi}{2}. \end{aligned}$$

shell method

$$V = 2\pi \int_0^1 x(x^2 + 1) dx = \frac{3\pi}{2}.$$

3. Find the volume of the solid formed by revolving the region bounded by $y = x^3 + x + 1$, $y = 1$, $x = 1$, $x = 1$ about the the line $x = 2$.

$$V = 2\pi \int_0^1 (2 - x)(x^3 + x + 1) dx = \frac{29\pi}{15}.$$

P.471

7.4 Arc length and surface of revolution

Arc length
Consider thr graph of the function $y = f(x)$ between the interval $[a, b]$.

Divide $[a, b]$ into $a = x_0 < x_1 < \cdots < x_n = b$.

$$\begin{aligned} l &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\triangle x_i)^2 + (\triangle y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{(1 + (\frac{\triangle y_i}{\triangle x_i})^2} \triangle x_i \end{aligned}$$

$\Rightarrow l = \lim_{\|\triangle\| \to 0} \sum_{i=1}^{n} \sqrt{(1 + (\frac{\triangle y_i}{\triangle x_i})^2} \triangle x_i = \int_a^b \sqrt{1 + (f'(x))^2} dx$

Example 1 Find the length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$. $(\frac{33}{16})$
area of surface of revolution

$$\begin{aligned} \triangle S_i = 2\pi r_i \triangle L_i &= 2\pi f(d_i) \sqrt{\triangle x_i^2 + \triangle y_i^2} \\ &= 2\pi f(d_i) \sqrt{(1 + (\frac{\triangle y_i}{\triangle x_i})^2} \triangle x_i. \end{aligned}$$

So we have the area of a surface of revolution to be

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

Example 1 Find the area of the surface formed by revolving the graph of $f(x) = x^3$ about the x-axis on the interval $[0, 1]$. $(\frac{\pi}{27}(10^{\frac{3}{2}} - 1))$

8 Technique of integration

8.1 Basic integration rules

Example 1 Find the integrals
(a). $\int \frac{4}{x^2+9} dx$ (b). $\int \frac{4x}{x^2+9} dx$ (c). $\int \frac{4x^2}{x^2+9} dx$

$sol.\ (a).\ \int \frac{4}{x^2+9}dx = 4\int \frac{1}{3^2+x^2}dx = \frac{4}{9}\int \frac{1}{1+(\frac{x}{3})^2}dx = \frac{4}{3}arctan\frac{x}{3} + C.$

$(b).\ u = x^2+9 \Rightarrow du = 2xdx.$

$$\int \frac{4x}{x^2+9}dx = \int \frac{2du}{u} = 2ln(x^{2+9}) + C.$$

$(c).\ \int \frac{4x^2}{x^2+9}dx = \int \frac{4(x^2+9)-36}{x^2+9}dx = \int (4-\frac{36}{x^2+9})dx = 4x - 12arctan\frac{x}{3} + C.$

2. $\int \frac{x+3}{\sqrt{4-x^2}}dx = \int \frac{x+3}{\sqrt{4}}dx - \int \frac{x+3}{\sqrt{x^2}}dx \quad u = 4-x^2 \quad du = -2xdx$

$= -\frac{1}{2}\int \frac{du}{u} + \frac{3}{2}\int \frac{dx}{\sqrt{1-(\frac{x}{2}^2)}} = -\sqrt{4-x^2} + 3arcsin\frac{x}{2} + C.$

3. $\int \frac{x^2}{\sqrt{16-x^6}}dx = \int \frac{x^2}{\sqrt{4^2-(x^3)^2}}dx = \frac{1}{4}\int \frac{x^2dx}{\sqrt{1-(\frac{x^3}{4})^2}} \quad u = \frac{x^3}{4} \quad du = \frac{3}{4}x^2dx$

$= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}dx = \frac{1}{3}arcsin\frac{x^3}{4} + C.$

4. $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx = \int (1-\frac{e^x}{1+e^x})dx = x - ln(1+e^x) + C.$

5. $\int cotxln(sinx)dx = \int udu = \frac{1}{2}u^2 + C = \frac{1}{2}(ln|sinx|)^2 + C.$

$u = lnsinx \quad du = \frac{cosx}{sinx}dx = cotxdx$

6. $\int tan^2 2xdx = \frac{1}{2}\int tan^2 udu = \frac{1}{2}(\int sec^2 u - \int du) \quad u = 2x \quad du = 2dx$

$= \frac{1}{2}tanu - \frac{u}{2} + C = \frac{1}{2}tan2x - x + C.$

$(i)\ \dfrac{1+x}{x^2+1} = \dfrac{1}{x^2+1} + \dfrac{x}{x^2+1}$

$(ii)\ \dfrac{1}{\sqrt{2x-x^2}} = \dfrac{1}{\sqrt{1-(x-1)^2}}$

$(iii)\ \dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1}$

$(iv)\ \dfrac{2x}{x^2+2x+1} = \dfrac{2x+2-2}{x^2+2x+1} = \dfrac{2x+2}{x^2+2x+1} + \dfrac{2}{(x+1)^2}$

$(v)\ \dfrac{1}{1+sinx} = \dfrac{1}{1+sinx}\dfrac{1-sinx}{1-sinx} = \dfrac{1-sinx}{1-sin^2x} = \dfrac{1-sinx}{cos^2x} = sec^2 - \dfrac{sinx}{cos^2x}$

8.2 Integration by parts

Consider the product rule

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u) = uv^{'} + vu^{'}$$

Apply integration on both sides, we have

$$uv = \int uv^{'} + \int vu^{'} = \int udv + \int vdu$$

$$\Rightarrow \int udv = uv - \int vdu \quad integration\ by\ parts\ formula.$$

Examples

$$1. \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

$u = x \quad du = dx$

$dv = e^x dx \quad v = e^x$

$$2. \int x^2 lnxsx = \frac{1}{3}x^3 lnx - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 lnx - \frac{1}{9}x^3 + C.$$

$u = lnx \quad du = \dfrac{1}{x}dx$

$dv = x^2 dx \quad v = \dfrac{1}{3}x^3$

3. $\int (arcsinx)dx = xarcsinx - \int \frac{x}{\sqrt{1-x^2}}dx \qquad u = arcsinx \quad du = \frac{dx}{\sqrt{1-x^2}}$

$dv = dx \qquad v = x$

$= xarcsinx + \frac{1}{2}\int \frac{du}{\sqrt{u}}dx$

$u = 1 - x^2 \quad du = -2xdx$

$= xarcsinx + \sqrt{1-x^2} + C.$

4. $\int x^2 sinxdx = -x^2cosx + 2\int xcosxdx \qquad u = x^2 \quad du = 2dx$

$dv = sinxdx \qquad v = -cosx$

$\int xcosxdx = sinx - \int sinxdx \qquad u = x \quad du = dx$

$dv = cosxdx \qquad v = sinx$

$= sinx + cosx + C.$

$\Rightarrow \int x^2 sinxdx = -x^2cosx + 2xsinx + 2cosx + C.$

5. $\int e^x sinxdx$

6. $\int sce^3xdx = scetanx - \int secxtan^2xdx \qquad u = secx \quad du = secxtanxdx$

$dv = sec^2x \qquad v = tanx$

$= secxtanx - \int secx(sec^2x - 1)dx$

$= secxtanx - \int sec^3xdx + \int secxdx$

$\int secxdx = ln|secx + tanx|$

$\Rightarrow \int sce^3xdx = \frac{1}{2}secxtanx + \frac{1}{2}ln|secx + tanx| + C.$

8.3 Trigonometric integrals

We will consider integral of the following forms:

$$\int sin^m x cos^n x dx \quad or \quad \int sec^m x tan^n x dx.$$

1. If the power of sine is odd and positive, i.e., $m = 2k+1$, $k \geq 0$.

$$\int sin^{2k+1} x cos^n x dx = \int (sin^2 x)^k cos^n x sin x dx = \int (1 - cos^2 x)^k cos^n x sin x dx.$$

Now set $u = cosx$.
2. If the power of cosine is odd and positive, i.e., $n = 2k+1$, $k \geq 0$.

$$\int sin^m x cos^{2k+1} x dx = \int sin^m x (cos^2 x)^k cos x dx = \int sin^m x (1 - sin^2 x)^k cos x dx.$$

Now set $u = sinx$.
3. If the power of both sine and cosine are even and nonnegative, make use of the identity

$$sin^2 x = \frac{1 - cos2x}{2} \qquad cos^2 x = \frac{1 + cos2x}{2}$$

to convert the power into odd power of the cosine. Now 2.
Examples

$$1. \int sin^3 x cos^4 x dx$$
$$2. \int \frac{cos^3 x}{sin x} dx$$
$$3. \int cos^4 x dx$$

1. If the power of secant is **even** and **positive**,

$$\int sec^{2k} x tan^n x dx = \int (sec^2 x)^{k-1} tan^n x sec^2 x dx$$
$$= \int (1 + tan^2 x)^{k-1} tan^n x sec^2 x dx.$$

Now let $u = tanx \ \ du = sec^2xdx$.
2. If the power of secant is **odd** and **positive**,

$$\int sec^m x tan^{2k+1} xdx = \int sec^{m-1}x(tan^2x)^k xsecxtanxdx$$
$$= \int sec^{m-1}x(sec^2x - 1)^k xsecxtanxdx.$$

Now let $u = secx \ \ du = secxtanxdx$.
3. If no secant power and the power of tangent is even and positive,

$$\int tan^n xdx = \int tan^{n-2}(sec^2x - 1)xdx.$$

4. If the integral is of the form $\int sec^m xdx$ where m is odd and positive, use integral by parts.
5. If npne of the above applies, convert the integral into *sine* and *cosine*.

Examples

1. $\int \frac{tan^3x}{\sqrt{secx}}dx = \int (secx)^{-\frac{1}{2}} tan^3xdx = \int (secx)^{-\frac{3}{2}} tan^2x \ secxtanxdx$
2. $\int sec^4 3x \ tan^3 3xdx = \int (sec^2 3x) tan^3 3x \ sec^2 3xdx$
3. $\int tan^4 xdx = \int tan^2x(sec^2x - 1)dx = \int tan^2xsec^2xdx - \int tan^2xdx$
   $= \int tan^2xsec^2xdx - \int (sec^2x - 1)dx$
4. $\int \frac{secx}{tan^2x}dx = \int \frac{1}{cosx}(\frac{cosx}{sinx})^2dx = \int \frac{cosx}{sin^2x}dx$
5. $\int sin5x \ cos4xdx = \frac{1}{2}\int (sinx + sin9x)dx$

8.4 Trigonometric substitution

1. For integrals involving $\sqrt{a^2 - u^2}$.
Let $u = a \ sin\theta$, then $\sqrt{a^2 - u^2} = a \ cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
2. For integrals involving $\sqrt{a^2 + u^2}$.
Let $u = a \ tan\theta$, then $\sqrt{a^2 + u^2} = a \ sec\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

3. For integrals involving $\sqrt{u^2 - a^2}$.
Let $u = a\ sec\theta$, then $\sqrt{u^2 - a^2} = \pm a\ tan\theta$, $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{2} \le \theta \le \pi$.

Examples

1. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}} \qquad (-\frac{\sqrt{9-x^2}}{9x} + C).$

2. $\displaystyle\int \frac{dx}{4x^2+1} \qquad u = 2x \quad du = 2x$

$$= \frac{1}{2}\int \frac{du}{u^2+1}$$

$u = tan\theta \quad du = sec^2\theta$

$$= \frac{1}{2}\int \frac{sec^{2\theta}}{sec\theta} d\theta = \frac{1}{2} ln|sec\theta + tan\theta| + C = \frac{1}{2} ln|\sqrt{4x^2+1} + 2x| + C.$$

3. $\displaystyle\int \frac{dx}{(x^2+1)^{\frac{3}{2}}} \qquad u = tan\theta \quad du = sec^2\theta d\theta$

$$= \int \frac{sec^2\theta}{sec^3\theta} = \int cos\theta d\theta = sin\theta + C = \frac{x}{\sqrt{x^2+1}}.$$

4. $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx \qquad x = \sqrt{3} sec\theta \quad dx = \sqrt{3} sec\theta\ tan\theta d\theta$

$\sqrt{x^2-3} = \sqrt{3} tan\theta$

$x = \sqrt{3}\ sec\theta = 1, \theta = 0,$

$x = 2sec\theta = \dfrac{2}{\sqrt{3}}, \theta = \dfrac{\pi}{6}$

$$\Rightarrow \int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3} tan\theta \sqrt{3}}{\sqrt{3} sec\theta} sec\theta\ tan\theta d\theta = \sqrt{3} \int_0^{\frac{\pi}{6}} tan^2\theta d\theta$$

$$= \sqrt{3} \int_0^{\frac{\pi}{6}} (sec^2 - 1\theta) d\theta = \sqrt{3}\ tan\theta|_0^{\frac{\pi}{6}} - \sqrt{3} \cdot \frac{\pi}{6} = \sqrt{3}\ (\frac{1}{\sqrt{3}} - \frac{\pi}{6}) = 1 - \frac{\sqrt{3}\pi}{6}$$

8.5 Partial Fraction

Transform a rational function into a simple one, so that we can apply the methods before. For example,

$$\int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-\frac{5}{2})^2-(\frac{1}{2})^2}dx \quad (a=\frac{1}{2})\ \ x-\frac{1}{2} = sec\theta,\ d\theta = sec\theta\ tan\theta d\theta$$
$$= 2\int \frac{sec\theta\ tan\theta d\theta}{tan^2\theta} = \int csc\theta d\theta = 2\ ln|csc\theta - cot\theta| + C = ln|\frac{x-3}{x-2}| + C.$$

$$x^2-5x+6 = (x-2)(x-3), \frac{1}{x^2-5x+6} = \frac{-1}{x-2} + \frac{1}{x-3}$$
$$\int \frac{1}{x^2-5x+6}dx = \int \frac{1}{x-3}dx - \int \frac{1}{x-2}dx = ln|\frac{x-3}{x-2}| + C.$$

Method of partial fractions

1. Divide if improper. If $deg\ N(x) \geq deg\ D(x)$, we divide into the form

$$\frac{N}{D} = polynomial + \frac{N_1(x)}{D(x)},$$

where $deg\ N_1 < deg\ D$.

2. Factor the denominator $D(x)$ completely into the $(px+q)^m$ and $(ax^2+bx+c)^n$, where ax^2+bx+c is irreducible.

3. Linear factors: for each factor of the form $(px+q)^m$, the partial fraction has the form

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}.$$

4. Quadratic factors: For each factor of the form $(ax^2+bx+c)^n$, the partial fraction has the form

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^m}.$$

Examples

1. $\displaystyle\int \frac{1}{x^2-5x+6}dx$

2. $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx \qquad fracAx + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \qquad A=6, B=-1, C=9.$

3. $\displaystyle\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx \qquad \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4} \qquad A=2\ (x=0),\ B=-1\ (x=1)$

$x=-1,\ x=2 \Rightarrow \begin{cases} -C+D=2 \\ 2C+D=8 \end{cases} \Rightarrow \begin{cases} C=2 \\ D=4. \end{cases}$

4. $\displaystyle\int \frac{8x^3+13x}{(x^2+2)^2}dx \qquad \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$

multiplying the common denominatior to get $A=8,\ B=0,\ C=-3,\ D=0.$

8.6 L$'$Hopital$'$s Rule

Indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$.

Theorem 8.3 f, g: diff. on (a,b) and continuous on $[a,b]$ s.t. $g'(x) \neq 0$ in (a,b), then $\exists$ a point $c \in (a,b)$ s.t. (Extended MVT)

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

proof. We may assume that $g(a) \neq g(b)$. Let

$$h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g(x).$$

Then

$$h(a) = \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)}, \quad h(b) = \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)}.$$

□

Now by Rolle$'$s thm $\exists$ a point $c \in (a,b)$ s.t. $h'(c)=0$ and hence

$$f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}g(c) = 0.$$

Theorem 8.4 L'Hopital's Rule f, g: diff. on (a,b) containing c, except possibly at c. Assume that $g^{'}(x) \neq 0$ in (a,b) except possibly at c. If the limit $\frac{f(x)}{g(x)}$ as x approaches c produces the form $\frac{0}{0}$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f^{'}(x)}{g^{'}(x)}$ provided the limit on the right exists. The results applies to $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, $\frac{-\infty}{-\infty}$.

proof. Consider the case $\lim_{x\to c^+} f(x)= 0$, $\lim_{x\to c^+} g(x)= 0$. The other cases are similar. So let

$$F(x) = \begin{cases} f(x) & x \neq c \\ 0 & x = c, \end{cases}$$

$$G(x) = \begin{cases} g(x) & x \neq c \\ 0 & x = c. \end{cases}$$

For any $c < x < b$, F and G are differentiable on $(c,x]$ and continuous on $[c,x]$. So we may apply Thm 8.3 to get that $\exists\ z \in (c,x)$ s.t.

$$\frac{f^{'}(z)}{g^{'}(z)} = \frac{F^{'}(z)}{G^{'}(z)} = \frac{F(x)-F(c)}{G(x)-G(c)} = \frac{F(x)}{G(x)} = \frac{f(x)}{g(x)}$$

Now let $x \to c^+$ then $z \to c^+$ since $c < z < x$. So we have $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{z\to c}\frac{f^{'}(z)}{g^{'}(z)}$. □

Example 1 $\lim_{x\to 0}\frac{e^{2x}-1}{x}$. (2)
Example 2 $\lim_{x\to\infty}\frac{lnx}{x}$.
Example 3 $\lim_{x\to-\infty}\frac{x^2}{e^{-x}}$.
Example 4 $\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty}\frac{e^{-x}}{\sqrt{x}}$.
Example 5 $\lim_{x\to\infty}(1+\frac{1}{x})^x$. (1^∞)
Let $y = \lim_{x\to\infty}(1+\frac{1}{x})^x \Rightarrow lny = \lim_{x\to\infty} xln(1+\frac{1}{x}) = \lim_{x\to\infty}\frac{ln(1+\frac{1}{x})}{\frac{1}{x}}$.
Example 6 $\lim_{x\to 0^+}(sinx)^x$.
Let $y = \lim_{x\to 0^+}(sinx)^x \Rightarrow lny = \lim_{x\to\infty}(xln(sinx)) = \lim_{x\to\infty}\frac{ln(sinx)}{\frac{1}{x}}$.
Example 7 $\lim_{x\to 1^+}\frac{1}{lnx} - \frac{1}{x-1} = \lim_{x\to 1^+}\frac{x-1-lnx}{(x-1)lnx}$. Apply L'Hopital twice.

8.7 Inproper Integral

1. Infinite limit of integration.

2. Infinite discontinuity.

1. $\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$.
2. $\int_{-\infty}^b f(x)dx := \lim_{a\to-\infty} \int_a^b f(x)dx$.
3. $\int_{-\infty}^\infty f(x)dx := \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$, c is any real number.

Examples 1. $\int_1^\infty \frac{dx}{x} =$
2. $\int_0^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx = \lim_{b\to\infty}(arctanx|_0^b) = \lim_{b\to\infty} arctanb = \frac{\pi}{2}$.
3. $\int_1^\infty (1-x)e^{-x}dx$
$u = 1-x \quad du = -dx$
$dv = e^{-x} \quad v = -e^{-x}$
$\int(1-x)e^{-x}dx = -(1-x)e^{-x} - \int e^{-x}dx = xe^{-x} + C$.
Now use L$'$H$\widehat{\text{o}}$pital Rule to find the limit. $(-\frac{1}{e})$
4. $\int_{-\infty}^\infty \frac{e^x}{1+e^x}dx = \int_{-\infty}^0 \frac{e^x}{1+e^x}dx + \int_0^\infty \frac{e^x}{1+e^x}dx$
$= \lim_{b\to-\infty}(arctane^x|_b^0) + \lim_{b\to\infty}(arctane^x|_0^b)$
$= \lim_{b\to-\infty}(\frac{\pi}{4} - arctan\ e^b) + \lim_{b\to\infty}(arctan\ e^b - \frac{\pi}{4})$
$= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$.

· Definition of improper integrals with infinite discontinuities.
1. If f is continuous on $[a,b)$ with an infinite discontinuity at b, then $\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$.
2. If f is continuous on $(a,b]$ with an infinite discontinuity at a, then $\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$.
3. If f is continuous on $[a,b]$, except for some c at which f has an infinite discontinuity, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Examples 1. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$
Since $\frac{1}{\sqrt[3]{x}}$ has an infinite discontinuity at $x = 0$,
$\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{b\to0^+}(\frac{3}{2}x^{\frac{1}{3}}|_b^1) = \lim_{b\to0^+} \frac{3}{2}(1 - b^{\frac{2}{3}}) = \frac{3}{2}$.
2. $\int_0^2 \frac{dx}{x^3}$
The integrand $\frac{1}{x^3}$ has an infinite discontinuity at $x = 0$,
$\int_0^2 \frac{dx}{x^3} = \lim_{b\to0^+}(-\frac{1}{2}x^{-2}|_b^2) = \lim_{b\to0^+}(-\frac{1}{8} + \frac{1}{2b^2}) = \infty$.
3. $\int_{-1}^2 \frac{dx}{x^3}$
As above the integrand $\frac{1}{x^3}$ has an infinite discontinuity at $x = 0$, so
$\int_0^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}$ diverges.

4. $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$
$= \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)} \quad u = \sqrt{x} \quad du = \frac{dx}{2\sqrt{x}}$
$= \lim_{b \to 0^+} (2arctan\sqrt{x}|_b^1) + \lim_{c \to \infty} (2arctan\sqrt{x}|_1^c)$
$= 2(\frac{\pi}{4}) - 0 + 2(\frac{\pi}{2} - \frac{\pi}{4}) = \pi.$