1 Limits

1.1

1.2 Finding limits graphically

Examples 1. $f(x) = \frac{x^3-1}{x-1}$, $x \neq 1$.
The behavior of $f(x)$ as x approximates 1.
. $\lim\limits_{x\to 1} f(x) = 3$.

2. $f(x) = \frac{x}{\sqrt{x+1}-1}$ $\lim\limits_{x\to 0} f(x) = 2$.
The behavior of $f(x)$ as $x = 0$.

3. Let $f(x) = \begin{cases} 1 & x \neq 2 \\ 0 & x = 2 \end{cases}$ $\lim\limits_{x\to 2} f(x) = 1$, but $f(x) = 0$.

Examples. (Limits that fail to exist)
1. $\lim\limits_{x\to 0} \frac{|x|}{x}$.
2. $\lim\limits_{x\to 0} \frac{1}{x^{(2)}}$.
3. $\lim\limits_{x\to 0} \frac{sin1}{x^{(2)}}$.

Formal definition of limit.
$\lim\limits_{x\to c} f(x) = L$. f becomes arbitrarily close to L
. as x approaches c.
ϵ-δ definition $| f(x) - L |< \epsilon$, $0 <| x - c |< \delta$
. for each ϵ, $\exists$ δ s.t.

1.3 Finding limits analytically

Theorem 1.1 b,c real numbers, n: positive integer.
(i) $\lim\limits_{x\to c} b = b$, (ii) $\lim\limits_{x\to c} x = c$, (iii) $\lim\limits_{x\to c} x^n = c^n$.

Theorem 1.2 f,g functions with limits $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = K$.
1. $\lim\limits_{x\to c} bf(x) = bL$.
2. $\lim\limits_{x\to c} f(x) \pm g(x) = L \pm K$.
3. $\lim\limits_{x\to c} f(x)g(x) = LK$.
4. $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ if $K \neq 0$.
5. $\lim\limits_{x\to c} (f(x))^n = L^n$.

Examples. $\lim\limits_{x\to 2} (4x^2 + 2)$
. $= \lim\limits_{x\to 2} 4x^2 + \lim\limits_{x\to 2} 2$
. $= 4 \cdot 2^2 + 2 = 18$.

Theorem 1.3
$\Rightarrow$ (i) If $p(x)$ is a polynomial, then $\lim\limits_{x\to c} p(x) = p(c)$.

Examples. $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$, $\lim\limits_{x\to -3}\frac{x^2+x-6}{x+3}$,
$lim\frac{\sqrt{x+1}-1}{x}$.

Theorem 1.8 (The squeeze theorem)
If $h(x) \le f(x) \le g(x)$ in an open interval containung c, except possibly at c and if $\lim\limits_{x\to c}h(x)=L=\lim\limits_{x\to c}g(x)$
then $\lim\limits_{x\to c}f(x) = L$.

Example 1. $\lim\limits_{x\to 0}\frac{sin\ x}{x} = 1$, $\lim\limits_{x\to 0}\frac{1-cos\ x}{x} = 0$.
$A = \frac{tan\theta}{2} \ge \frac{\theta}{2} \ge \frac{sin\theta}{2}$
$\frac{1}{cos\theta} \ge \frac{\theta}{sin\theta} \ge 1$
$cos\theta \le \frac{sin\theta}{\theta} \le 1$
Now apply squeeze thm.

2. $\lim\limits_{x\to 0}\frac{tan\ x}{x}=\lim\limits_{x\to 0}\frac{sin\ x}{x}\frac{1}{cos\ x}=\lim\limits_{x\to 0}\frac{sin\ x}{x}\lim\limits_{x\to 0}\frac{1}{cos\ x} = 1$.

3. $\lim\limits_{x\to 0}\frac{sin\ 4x}{x}=4\lim\limits_{x\to 0}\frac{sin\ 4x}{4x}=4\lim\limits_{x\to 0}\frac{sin\ y}{y} = 4$.

1.4 Continuity and one-side limits

Definition: A function f is *continuous at c* if $f(c)$ is defined, $\lim\limits_{x\to c}f(x)$ exists and $\lim\limits_{x\to c}f(x) = f(c)$.
The function f is *continuous on an open interval* (a, b) if it is continuous at each point in (a, b).

Discontinuity: jump, infinity, not defined.

One-side limits
limit from the right $\lim\limits_{x\to c^+}f(x) = L$,
limit from the left $\lim\limits_{x\to c^-}f(x) = L$.

Examples. greatest integer function $f(x) = [x]$.
$\lim\limits_{x\to 0^-}[x] = -1$, $\lim\limits_{x\to 0^+}[x] = 0$.

Theorem 1.10 f:function, c, L: real numbers.
Then $\lim\limits_{x\to c}f(x) = L$ iff $\lim\limits_{x\to c^+}f(x) = L$ and $\lim\limits_{x\to c^-}f(x) = L$.

Definition: A function f is *continuous on the closed interval* $[a, b]$ if f is continuous on (a, b) and $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$.

Examples. $f(x) = \sqrt{1 - x^2}$.
$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1)$,
$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1)$.
$f(x)$ is continuous on $[-1, 1]$.

Theorem 1.11 b: real number. f, g are continuous at $x = c$, then the following functions are continuous:
(a) bf.
(b) $f \pm g$.
(c) fg.
(d) $\frac{f}{g}$ if $g(c) \neq 0$.

Theorem 1.12 If g is continuous at $x = c$, f is continuous at $g(c)$, then the composite $(f \circ g)(x) = f(g(x))$ is continuous at $x = c$.

Examples. Test for continuity.
(a) f(x)=tan x.(not defined at $x = \frac{\pi}{2} + n\pi, n \in \mathbb{N}$) (b) $f(x) = \begin{cases} sin\ \frac{1}{x} & x \neq 0 \\ 0 & x = 2. \end{cases}$
(c) $f(x) = xsin\ \frac{1}{x}$, $-\mid x \mid \leq xsin\frac{1}{x} \leq \mid x \mid$.
Theorem 1.13 (Intermediate value theorem)
If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ s.t. $f(c) = k$.

Examples. $f(x) = x^3 + 2x - 1$. $f(x)$ has a root in $[a, b]$.

1.5 Infinite limits

Definition: f: function defined in an open interval containing c(except possibly at c).
$\lim_{x \to c} f(x) = \infty$
means that for each $M > 0$, $\exists \delta > 0$ s.t. $f(x) > M$ whenever $0 < \mid x - c \mid < \delta$.
$\lim_{x \to c} f(x) = -\infty$

means that for each $N > 0$, $\exists\delta > 0$ s.t. $f(x) < M$ whenever $0 <\mid x - c \mid< \delta$.
$\underset{x\to c^-}{lim} f(x) = \infty$ means that $f(x) > M$ for $c - \delta < x < c$.
In each of the above cases, we say $x = c$ is a vertical asymptate.

Theorem 1.14 If $h(x) = \frac{f(x)}{g(x)}$ where $f(x)$, $g(x)$ are continuous at $x = c$, with $f(c) \neq 0$ and $g(c) \neq 0$, then $h(c)$ has a vertical asymptate at $x = c$.

Examples. (a) $f(x) = \frac{1}{2(x+1)}$. (b) $f(x) = \frac{x^2+1}{x^2-1)}$. (c) $f(x) = cot\ x$. (d) $f(x) = \frac{x^2+2x-8}{x^2-4}$.

Theorem 1.15 If $\underset{x\to c}{lim} f(x) = \infty$, $\underset{x\to c}{lim} g(x) = L$.
1. $\underset{x\to c}{lim} (f(x) \pm g(x)) = \infty$.
2. $\underset{x\to c}{lim} f(x)g(x) = \begin{cases} \infty & \text{if } L > 0, \\ -\infty & \text{if } L < 0. \end{cases}$
3. $\underset{x\to c}{lim} (\frac{g(x)}{f(x)}) = 0$.

2 Differentiation

2.1 The derivative and the tangent line problem

Definition: If f is defined on an open interval containing c, and if the limit
$\underset{\triangle x\to c}{lim} \frac{\triangle y}{\triangle x} = \underset{\triangle x\to c}{lim} \frac{f(c+\triangle x)-f(c)}{\triangle x} = m$
exists, then the line through $(c, f(c))$ with shape m is the *tangent line* to the graph of f at $(c, f(c))$.

The *derivative* of f at x is defined by
$f^{'}(x) = \underset{\triangle x\to c}{lim} \frac{f(c+\triangle x)-f(c)}{\triangle x}$
provided the limit exists. Forr all x for which the limit exists, $f^{'}$ is a function of x.

Remark: The process of finding the derivative of a finction is called *differentiation*. A function f is *differentiable* at x if $f^{'}(x) =$ exists. f is called differentiable on (a, b), if it is differentiable at each point x in (a, b).

Notation: $f^{'}(x)$, $\frac{dy}{dx}$, $y^{'}$, $\frac{d}{dx}f$.

Examples. (a) $f(x) = x^2 + 1$, $f(x) = \sqrt{x}$. ($f^{'}(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$)

Differentiability:
Not differentiable if (1) $f^{'}(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \neq \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$
or (2) $f^{'}(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \pm\infty$.
Examples. (a) $f(x) =\mid x-2 \mid$, (b) $f(x) = x^{\frac{1}{3}}$.

Theorem 2.1 If f is differentatiable at $x = c$, then f is continuous at $x = c$.

proof. $\lim_{x \to c} f(x) - f(c) = \lim_{x \to c} (x-c)\frac{f(x)-f(c)}{x-c} = \lim_{x \to c}(x-c) \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$
$= \lim_{x \to c}(x-c) \cdot f^{'}(c) = 0$.
This means $\lim_{x \to c} f(x) = f(c)$,i.e. $f(x)$ is continuous at c. □

2.2 Bsic differentiation rule

Theorem 2.2 If c is a real number, then
$\frac{d}{dx}c = 0$.

Theorem 2.3 If n is any rational number, then $f(x) = x^n$ is differentiable and $f^{'}(x) = nx^{n-1}$.
In particular, we have $\frac{d}{dx}x = 1$.

Examples. Ex2. p109.

Theorem 2.4 If f is differentiable and c is any real number, then
$\frac{d}{dx}(cf(x)) = cf^{'}(x)$.

Theorem 2.5 If $f(x)$ and $g(x)$ are differentiable, then
$f \pm g$ are differentiable and
$\frac{d}{dx}(f \pm g)(x) = f^{'}(x) \pm g^{'}(x)$.

Examples. (a) $f(x) = \frac{1}{x^2}$, (b) $f(x) = x^3 - 4x + 5$.

Theorem 2.6 $\frac{d}{dx}(sin\ x) = cos\ x$, $\frac{d}{dx}(cos\ x) = -sin\ x$.

$sin(x + y) = sin\ xcos\ y + cos\ xsin\ y$.

2.3 product and quotient rules

Theorem 2.7 (product rule)
If $f(x)$ and $g(x)$ are differentiable, then
$\frac{d}{dx}(f(x)g(x)) = f^{'}(x)g(x) + f(x)g(x)^{'}$.

Theorem 2.8 (quotient rule)
If $f(x)$ and $g(x)$ are differentiable with $g(x) \neq 0$,
then $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f^{'}(x)g(x)-f(x)g(x)^{'}}{g(x)^2}$.

Examples. Fing the derivatives
1. $h(x) = (3x - 2x^2)(5 + 4x)$.
2. $y = \frac{5x-2}{x^2+1}$.
3. $y = 3x^2 sin\ x$.
4. The tangent line of $f(x) = \frac{3-\frac{1}{x}}{x+5}$ at $(-1, 1)$.
5. $y = \frac{1-cos\ x}{sin\ x} = csc\ x - cot\ x$.

High order derivatives

1^{st} order	$y^{'}$	$f^{'}(x)$	$\frac{dy}{dx}$	$\frac{df}{dx}$
2^{nd} order	$y^{''}$	$f^{''}(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f$
$\vdots$				
n^{th} order	y^n	$f^n(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}f$

Theorem 2.8 $\frac{d}{dx}(tan\ x) = sec^2\ x$, $\frac{d}{dx}(cot\ x) = -csc^2\ x$.
$\frac{d}{dx}(sec\ x) = sec\ xtan\ x$, $\frac{d}{dx}(csc\ x) = -csc\ xcot\ x$.

2.4 Chain rule

Theorem 2.10 If $y = f(x)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x then $y = f(g(x))$ is a differentiable function of x and

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
or $\frac{d}{dx}f(g(x)) = f^{'}(g(x)) \cdot g^{'}(x)$.

Examples. Fing $\frac{dy}{dx}$ if
(1) $y = (x+1)^3$.
(2) general power rule $y = (u(x))^u$, n rational, $u(x)$ is differentiable then $\frac{dy}{dx} = nu(x)^{n-1}\frac{du}{dx}$.
(3) $f(x) = \sqrt[3]{(x^2-1)^2}$.
(4) $g(t) = \frac{-7}{(2t-3)^2}$.
(5) $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$.
(6) $y = (\frac{3x-1}{x^2+3})^2$.
(7) $\frac{d}{dx}(sin\ u) = cos\ u\frac{du}{dx}$, $\frac{d}{dx}(cos\ u) = -sin\ u\frac{du}{dx}$.
$\frac{d}{dx}(tan\ u) = sec^2\ u\frac{du}{dx}$, $\frac{d}{dx}(cot\ u) = -csc^2\ u\frac{du}{dx}$.
(8) $y = cos^2 x$.
(9) $f(t) = sin^3 4t$.

2.5 Implicit differentiation

If the function y of x is given implicitly by
$f(x, y) = 0$,
we can still find $\frac{dy}{dx}$ in terms of $f(x, y)$ and the chain rule.

Examples. 2, 4, 5, 6, 7.

3 Application of differentiation

3.1 Extreme on an interval

Definition:(extreme)
f: function defined on an interval I containing c.
1. $f(c)$ is the **minimum** of f on I if $f(c) \leq f(x)$, $\forall x \in I$.
2. $f(c)$ is the **maximum** of f on I if $f(c) \geq f(x)$, $\forall x \in I$.
They are also called the *extreme values*, *absolute min* or *absolute max*.

Theorem 2.10 (extreme value theorem)
If $y = f(x)$ is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Definition:(relative extrema)
1. If $\exists$ open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or f has a relative maximum at $(c, f(c))$.
2. If $\exists$ open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or f has a relative minimum at $(c, f(c))$.

Definition:(critical numbers)
Let f be defined on c. If $f^{'}(c) = 0$ or f is **Not** differentiable at c, then c is a critical number of f.
Theorem 2.10 If f has a relative minimum or relative maximum at c, then c is a critical number of f.

proof. (i) If f is Not differentiable, then by definition c is a critical number.
(ii) If f is differentiable at $x = c$, $f^{'}(x)$ is > 0, $= 0$, < 0. If $f^{'}(c) > 0$, i.e. $f^{'}(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$. This implies that $\frac{f(x)-f(c)}{x-c} > 0$ in a neighborhood of c, $x \neq c$.
left of c, $x < c$, and $f(x) < f(c) \Rightarrow f(c)$ is Not a relative minimum,
right of c, $x > c$, and $f(x) > f(c) \Rightarrow f(c)$ is Not a relative maximum.
So, $f^{'}(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum. □

Finding extreme on a closed interval $[a, b]$
1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number.
3. Evaluate f at the endpoints a and b.
4. The least of these is the minimum and the largest is the maximum.

Examples. Find the extrema of $f(x) = 2x - 3x^{\frac{2}{3}}$ on $[-1, 3]$. p.168.

3.2 Rolle$^{'}$s theorem and the MVT

Theorem 3.3 (Rolle) f: continuous on $[a, b]$ and differentiable on (a, b).
If $f(a) = f(b)$, then $\exists\ c \in (a, b)$ s.t. $f^{'}(c) = 0$.

proof. (a) If $f(x) = d = f(a) = f(b)$, f is const. on the interval $[a, b]$, $f^{'}(x) = 0$ for any $x \in (a, b)$.
(b) If $f(x) > d$ for some $x \in (a, b)$. By extreme value theorem, f has a maximum at some point c in the interval. Since $f(c) > d$, the maximum is **NOT** at the endpoints. This implies that $f(c)$ is relative maximum, by theorem $f^c = 0$, since f is differentiable at c.
(c) If $f(x) < d$ for some $x \in (a, b)$, we get a relative minimum by the argument above. □

Theorem 3.4 (MVT) If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists\ c \in (a, b)$ s.t. $f^{'}(c) = \frac{f(b)-f(c)}{b-a}$.

proof. Define the function $g(x)$ by
$g(x) = f(x) - \frac{f(b)-f(c)}{b-a}(x - a) - f(a)$.
Then $g(a) = 0 = g(b)$, g is continuous on $[a, b]$ and differentiable on (a, b).
By Rolle's theorem $\exists c \in (a, b)$ s.t. $g^{'}(c) = 0$. Now
$0 = g^{'}(c) = f^{'}(c) = \frac{f(b)-f(c)}{b-a}$.
i.e. $f^{'}(c) = \frac{f(b)-f(c)}{b-a}$. □

EX 3. p175.

3.3 Increasing and decreasing functions and the 1^{st} derivative test

Definition: A function f is
increasing on an interval I if for any x_1, $x_2 \in I$,
$x_1 < x_2$ implies that $f(x_1) < f(x_2)$.
decreasing on an interval I if for any x_1, $x_2 \in I$,
$x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

Theorem 3.5 Test for increasing and decreasing.
f: continuous on $[a, b]$, differentiable on (a, b).
1. If $f^{'}(x) > 0 \forall x \in I$, then f is increasing on $[a, b]$.
2. If $f^{'}(x) < 0 \forall x \in I$, then f is decreasing on $[a, b]$.
1. If $f^{'}(x) = 0 \forall x \in I$, then f is constant on $[a, b]$.

proof. If $f^{'}(x) > 0 \forall x \in I$. For any $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. By MVT $\exists x_1 < c < x_2$ s.t. $f^{'}(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$.
$\Rightarrow f(x_2) > f(x_1)$. i.e. f is increasing on (a,b). □

Examples 1 $p180$.

Finding functions on which a function is increasing or decreasing.
1. Locate the critical numbers of f in (a,b), and use these numbers to determine the test intervals.
2. In each of the intervals, determine the sign of $f^{'}(x)$ at one testnumber.
3. Use the above Theorem 3.5.

Theorem 3.6 c is a critical number of f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then
1. If $f^{'}(x)$ changes from negative to positive at c, then f has a relative minimum at $(c, f(c))$.
2. If $f^{'}(x)$ changes from positive to negative at c, then f has a relative maximum at $(c, f(c))$.
3. If $f^{'}(x)$ is positive on both sides of c or is negative on sides of c, then f neither a relative minimum nor a relative minimum.

Examples 3, 4 p183.

3.4 Concavity and 2^{nd} derivative test

Definition: f: differentiable on I.,
The graph of f is **concave upward** on I if $f^{'}(x)$ is increasing on I,
. **concave downward** on I if $f^{'}(x)$ is decreasing on I.

The above are equivalent to the following:
f is concave upward on $I \Leftrightarrow$ the graph of f lies above of its tangent lines on I.
f is concave downward on $I \Leftrightarrow$ the graph of f lies below of its tangent lines on I.

Theorem 3.7(Test for concavity)
f function on I whose 2^{nd} derivative exists on I.
1.If $f^{''}(x) > 0$ for all $x \in I$, then f is **concave upward** on I.
2.If $f^{''}(x) < 0$ for all $x \in I$, then f is concave downward on I.

Definition: f: continuous on open interval I, $c \in I$.
If the graph of f has a tangent line at the point $(c, f(c))$, then the point $(c, f(c))$ is a **point of inflection** of f is the concavity of f changes from upward to downward(or downward to upward).

Theorem 3.8 If $(c, f(c))$ is a point of inflection, then either $f^{''}(x) = 0$ or $f^{''}$ does Not exist at $x = c$.

Examples. $f(x) = x^4 - 4x^3$. Find the points of inflection.
$f^{'}(x) > 0 = 4x^3 - 12x^2$
$f^{''}(x) = 12x^2 - 24x = 12x(x-2)$
$f^{''}(x) = 0 \Rightarrow x = 0, 2$

.	-$\infty< x < 0$	$0 < x < 2$	$2 < x < \infty$
test value	$x = -1$	$x = 1$	$x = 3$
.	$f^{''}(-1) > 0$	$f^{''}(1) < 0$	$f^{''}(3) > 0$
.	-upward	downward	upward

Theorem 3.8 2^{nd} derivative test
f: a function s.t. $f^{'}(c) = 0$ and $f^{''}(x)$ exists on an open interval containing c.
1. If $f^{''}(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f^{''}(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f^{''}(c) = 0$, the test fails. That is, f may have a relative maximum, relative minimum or neither. In this case, we should use 1^{st} derivative test.

3.5 Limits and infinity

Definition: L: a real number.
1. $\lim_{x\to\infty} f(x) = L$ means that for every $\epsilon > 0$, $\exists M > 0$ s.t. $\mid f(x) - L \mid < \epsilon$ whenever $x > M$.
2. $\lim_{x\to-\infty} f(x) = L$ means that for every $\epsilon > 0$, $\exists N < 0$ s.t. $\mid f(x) - L \mid < \epsilon$ whenever $x < N$.

In both cases, we say that $y = L$ is a **horizontal asymptote** for the function $f(x)$ as $x \to \infty$.

Theorem 3.8 Limits at infinity.
IF $t > 0$ rational number, c is any real number, then
$\underset{x\to\infty}{lim} \frac{c}{x^r} = 0.$
If x^r is defined for $x < 0$, then
$\underset{x\to-\infty}{lim} \frac{c}{x^r} = 0.$

Ex 3, 4, 5.

If $\underset{x\to\infty}{lim} f(x)$, $\underset{x\to\infty}{lim} g(x)$ exists, then
$\underset{x\to\infty}{lim}(f(x) + g(x))=\underset{x\to\infty}{lim} f(x)+\underset{x\to\infty}{lim} g(x),$
$\underset{x\to\infty}{lim}(f(x)g(x))=\underset{x\to\infty}{lim} f(x)\cdot\underset{x\to\infty}{lim} g(x).$

Definition: 1.$\underset{x\to\infty}{lim} f(x) = \infty$ means $f(x) > M$ whenever $x > N$.

3.6 A summary and sketching

Guide lines:
1. Determine the domain and range of the function.
2. Deter the intercepts, asymptotes and symmetry of the graph.
3. Locate the x-value for which $f^{'}(x)$ and $f^{''}(x)$ either are zero or do Not exist. Use these results to determine relative extreme and points of inflection.

Examples 1, 2, 4, 5.

3.7

3.8

3.9 Differentials

The tangent line at $(c, f(c))$ is $y = f(c) + f^{'}(c)(x = c)$ when $x - c - \triangle x$ is vert small, the change $\triangle y = f(c + \triangle x) - f(c)$can be approximated by $\triangle y = f(c + \triangle x) - f(c) \approx f^{'}(c)\triangle x$. (1)

When $\triangle x \to 0$, denote it by dx, we then define the differential to $dy = f^{'}(x)dx$.

Examples 2, before 4, 5.

(1)$\Rightarrow f(x+\triangle x) \simeq f(x) + dy = f(x) + f^{'}(x)\triangle x$.