§13.5

⑧ $w = x\sqrt{y^2+z^2}$, $x = \frac{1}{t}$, $y = e^{-t}\cos t$, $z = e^{-t}\sin t$ $\Rightarrow \sqrt{y^2+z^2} = e^{-t}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \sqrt{y^2+z^2}(-t^{-2}) + \frac{xy}{\sqrt{y^2+z^2}} e^{-t}(-\cos t - \sin t) + \frac{xz}{\sqrt{y^2+z^2}} e^{-t}(-\sin t + \cos t)$$

$$= -e^{-t}t^{-2} - xy(\cos t + \sin t) + xy(-\sin t + \cos t)$$

$$= -e^{-t}t^{-2} - x(y\cos t + y\sin t + z\sin t - z\cos t)$$

㉔

$w = \cos(2x+3y)$, $x = r^2 st$, $y = s^2 tu$

$$\therefore \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = -4rst\sin(2x+3y)$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = -3s^2t\sin(2x+3y)$$

㉞ $x^2+y^2+z^2-xy-yz-xz=1$

$\therefore F(x,y,z) = x^2+y^2+z^2-xy-yz-xz-1=0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2x-y-z}{2z-y-x} = \frac{2x-y-z}{x+y-2z} \quad ; \quad \frac{\partial z}{\partial y} = \frac{2y-x-z}{x+y-2z}$$

㊻ $\therefore \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

$$+ \frac{1}{r^2}\left(\left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)r^2\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(52)

let $u = x^2 + y^2 \Rightarrow z = f(x^2+y^2) = f(u)$

$\therefore \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = 2x\frac{dz}{du}$

$\frac{\partial z}{\partial y} = 2y\frac{dz}{du}$

$\therefore y\left(\frac{\partial z}{\partial x}\right) - x\left(\frac{\partial z}{\partial y}\right) = 2xy - 2xy = 0$

(60)

$\because \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$

$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$

$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(-r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y}\right)$

$= \cos\theta\frac{\partial v}{\partial y} - \sin\theta\frac{\partial v}{\partial x}$

$= \cos\theta\frac{\partial u}{\partial x} - \sin\theta\left(-\frac{\partial u}{\partial y}\right)$

$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$

$= \frac{\partial u}{\partial r}$

$\because \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial v}{\partial y} + \sin\theta\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$

$\therefore -\frac{1}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial u}{\partial x} - \cos\theta\frac{\partial u}{\partial y}$

$= \sin\theta\frac{\partial v}{\partial y} - \cos\theta\left(-\frac{\partial v}{\partial x}\right)$

$= \frac{\partial v}{\partial r}$

§13.6

(19) f(

∴ u

f

f

f

∴ D_u

(26) f(

∴ u

f_x

f_y

f_z

∴ D_u

(30) f(

∴ u =

f_x(

f_y(

∴ $D_u f$

§13.6

(19) $f(x,y,z) = x^2y^3z^4$; $P(3,-2,1)$, $v = i+j+k$

$\because u = \frac{v}{|v|} = \frac{i+j+k}{\sqrt{1^2+1^2+1^2}} = \frac{\sqrt{3}}{3}(i+j+k)$

$f_x(x,y,z) = 2xy^3z^4$

$f_y(x,y,z) = 3x^2y^2z^4$

$f_z(x,y,z) = 4x^2y^3z^3$

$\therefore D_u f(3,-2,1) = 2\cdot 3\cdot(-2)^3(1)^4\left(\frac{\sqrt{3}}{3}\right) + 3\cdot 3^2(-2)^2 1^4\left(\frac{\sqrt{3}}{3}\right)$
$+ 4\cdot 3^2(-2)^3 1^3\left(\frac{\sqrt{3}}{3}\right)$
$= -76\sqrt{3}$.

(26) $f(x,y,z) = e^x(2\cos y + 3\sin z)$; $P(1, \frac{\pi}{6}, \frac{\pi}{6})$, $v = 2i - j + 3k$

$\because u = \frac{v}{|v|} = \frac{2i-j+3k}{\sqrt{2^2+(-1)^2+3^2}} = \frac{\sqrt{14}}{14}(2i-j+3k)$

$f_x(x,y,z) = e^x(2\cos y + 3\sin z)$

$f_y(x,y,z) = -2e^x \sin y$

$f_z(x,y,z) = 3e^x \cos z$

$\therefore D_u f(1, \frac{\pi}{6}, \frac{\pi}{6}) = e\left(2\cdot\frac{\sqrt{3}}{2} + 3\cdot\frac{1}{2}\right)\frac{\sqrt{14}}{7} - 2e\cdot\frac{1}{2}\left(-\frac{\sqrt{14}}{14}\right) + 3e\cdot\frac{\sqrt{3}}{2}\cdot\frac{3\sqrt{14}}{14}$
$= \frac{\sqrt{14}}{28}(13\sqrt{3}+8)e$.

(30) $f(x,y) = xe^{-y}$, $P(2,0)$, $Q(-1,2)$

$\because u = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{-3i+2j}{\sqrt{(-3)^2+2^2}} = -\frac{3\sqrt{13}}{13}i + \frac{2\sqrt{13}}{13}j$

$f_x(x,y) = e^{-y}$

$f_y(x,y) = -xe^{-y}$

$\therefore D_u f(2,0) = f_x(2,0)\left(-\frac{3\sqrt{13}}{13}\right) + f_y(2,0)\left(\frac{2\sqrt{13}}{13}\right)$
$= -\frac{7\sqrt{13}}{13}$.

(35) $f(x,y,z) = x^3 + 2xz + 2yz^2 + z^3$; $P(-1, 3, 2)$

$f_x(x,y,z) = 3x^2 + 2z$

$f_y(x,y,z) = 2z^2$

$f_z(x,y,z) = 2x + 4yz + 3z^2$

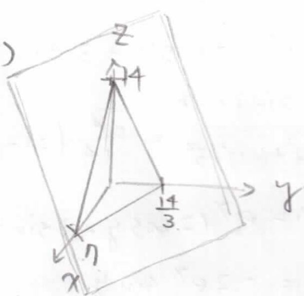
$\therefore \nabla f(-1,3,2) = f_x(-1,3,2)\,i + f_y(-1,3,2)\,j + f_z(-1,3,2)\,k$

$= 7i + 8j + 34k$

and $|\nabla f(-1,3,2)| = \sqrt{7^2+8^2+34^2} = 3\sqrt{141}$

§ 13.7.

(12) $F(x,y,z) = 2x + 3y + z$, $P(2,3,1)$



$\therefore F(2,3,1) = 14$

$\therefore 2x + 3y + z = 14$

$\nabla F(2,3,1) = 2i + 3j + k$

(23) $xz^2 + yx^2 + y^2 - 2x + 3y + 6 = 0$, $P(-2,1,3)$

$\nabla F(-2,1,3) = 3(i + 3j - 4k)$

$\therefore (x+2) + 3(y-1) - 4(z-3) = 0$

$\Rightarrow x + 3y - 4z = -11$ (tangent plant)

$\therefore$ the normal line passing through $(-2,1,3)$ are

$\frac{x+2}{1} = \frac{y-1}{3} = \frac{z-4}{-4} \Rightarrow x+2 = \frac{y-1}{3} = \frac{z-3}{-4}$

(34) $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$\nabla F(x_0,y_0,z_0) = \frac{2x_0}{a^2}i + \frac{2y_0}{b^2}j + \frac{2z_0}{c^2}k$

$\therefore$ equation of the tangent plane at (x_0, y_0, z_0)

is $\frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + \frac{2z_0}{c^2}(z-z_0) = 0$

$\Rightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 0$

But (x_0, y_0, z_0) lies on the hyperboloid

$\therefore$ the expression in parenthess is equal to 1

and we have $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$

(37) F(
the
poi

(42) F(
G(

(44) ∇
∇

(37) $F(x, y, z) = x^2 + y^2 + z^2 - 14$

the normal to the tangent plane to the sphere at the point (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = 2x_0 i + 2y_0 j + 2z_0 k$.

$\because$ $i + 2j + 3k$ is normal of $x + 2y + 3z = 12$.

$\therefore$ $x_0 i + y_0 j + z_0 k = c(i + 2j + 3k)$, c is constant.

$\Rightarrow x_0 = c,\ y_0 = 2c,\ z_0 = 3c$.

$\Rightarrow c^2 + (2c)^2 + (3c)^2 = 14$

$\Rightarrow c = \pm 1$

$\therefore$ $(-1, -2, -3)$ and $(1, 2, 3)$

(42) $F(x, y, z) = x^2 + y^2 + z^2 - 17$

$G(x, y, z) = 2x^2 - y + 2z^2 + z$

$\because$ $F(x, y, z) = 0$ at $(1, 4, 0)$

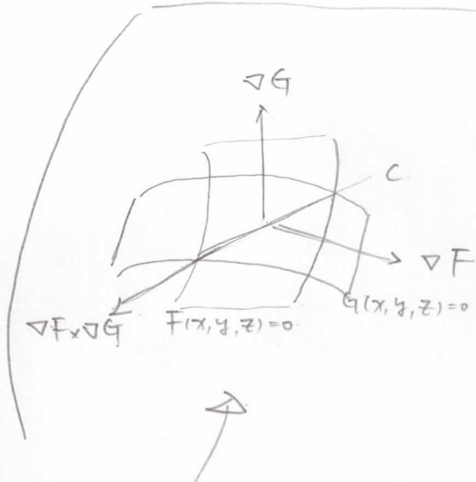
$\Rightarrow v = \nabla F(1, 4, 0) = 2i + 8j$

$\because$ $G(x, y, z) = 0$ at $(1, 4, 0)$

$\Rightarrow w = \nabla G(1, 4, 0) = 4i - j$

$\therefore$ $v \cdot w = (2i + 8j)(4i - j) = 0$

$\therefore$ v and w are orthogonal.



(44) $\nabla F(x_0, y_0, z_0)$ is normal to $F(x, y, z) = 0$ at P

$\nabla G(x_0, y_0, z_0)$ is normal to $G(x, y, z) = 0$ at P

$\therefore$ $\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$ is parallel to the tangent line to C at P.