4.2 Integration by Substitution

12. For $I = \int x^2 \left(2x^3 - 1\right)^{-4} dx$, let $u = 2x^3 - 1 \Rightarrow du = 6x^2\,dx \Rightarrow x^2\,dx = \frac{1}{6}\,du$. Then $I = \frac{1}{6}\int u^{-4}\,du = \frac{1}{6}\left(-\frac{1}{3}u^{-3}\right) + C = -\frac{1}{18}\left(2x^3 - 1\right)^{-3} + C$.

18. For $I = \int x^{-1/3}\sqrt{x^{2/3} - 1}\,dx$, let $u = x^{2/3} - 1 \Rightarrow du = \frac{2}{3}x^{-1/3}\,dx \Rightarrow x^{-1/3}\,dx = \frac{3}{2}\,du$. Then $I = \frac{3}{2}\int u^{1/2}\,du = \frac{3}{2}\left(\frac{2}{3}u^{3/2}\right) + C = \sqrt{\left(x^{2/3} - 1\right)^3} + C$.

30. For $I = \int x^2 \sec^2 x^3\,dx$, let $u = x^3 \Rightarrow du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{1}{3}\,du$. Then $I = \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan x^3 + C$.

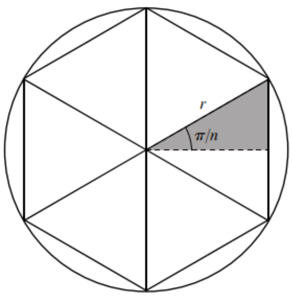
40. For $I = \int \csc^2 x\,(\cot x - 1)^3\,dx$, let $u = \cot x - 1 \Rightarrow du = -\csc^2 x\,dx$. Then $I = -\int u^3\,du = -\frac{1}{4}u^4 + C = -\frac{1}{4}(\cot x - 1)^4 + C$.

4.3 Area

33. $\sum_{k=1}^{10} k\,(k-2) = \sum_{k=1}^{10} k^2 - 2\sum_{k=1}^{10} k = \dfrac{10\,(10+1)\,(2\cdot 10+1)}{6} - \dfrac{2\,(10)\,(11)}{2} = 275$

41.
$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}+2\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\left(\frac{3}{n}\right)\left(\frac{1}{n}\sum_{k=1}^{n} k + 2\sum_{k=1}^{n} 1\right) = \lim_{n\to\infty}\left(\frac{3}{n}\right)\left[\frac{1}{n}\cdot\frac{n\,(n+1)}{2} + 2n\right]$$
$$= \lim_{n\to\infty}\left[\frac{3}{2}\left(1+\frac{1}{n}\right)+6\right] = \frac{15}{2}$$

57. a. The area of the shaded triangle is $\frac{1}{2}\,(\text{base})\,(\text{height}) = \frac{1}{2}\left(r\cos\frac{\pi}{n}\right)\left(r\sin\frac{\pi}{n}\right)$, so the area of each isosceles triangle is $2\cdot\frac{1}{2}r^2\cos\frac{\pi}{n}\sin\frac{\pi}{n} = \frac{1}{2}r^2\sin\frac{2\pi}{n}$. Therefore, $A_n = \frac{1}{2}r^2 n\sin\frac{2\pi}{n}$.



b. $A = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{1}{2}r^2 n\sin\frac{2\pi}{n} = \frac{1}{2}r^2\lim_{n\to\infty} n\left(\frac{2\pi}{2\pi}\right)\sin\frac{2\pi}{n}$

$$= \frac{1}{2}r^2\,(2\pi)\lim_{n\to\infty}\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi r^2$$

58. a. Refer to the figure in Exercise 57. The length of each side of the isosceles triangle is $2r\sin\frac{\pi}{n}$, so the perimeter of the polygon is $C_n = 2nr\sin\frac{\pi}{n}$.

b. $C = \lim_{n\to\infty} C_n = \lim_{n\to\infty} 2nr\sin\frac{\pi}{n} = 2r\lim_{n\to\infty} n\left(\frac{\pi}{\pi}\right)\sin\frac{\pi}{n} = 2\pi r\lim_{n\to\infty}\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 2\pi r$