9.8

4. $f(x) = e^{-2x}$, $f'(x) = -2e^{-2x}$, $f''(x) = (-2)^2 e^{-2x}$, ..., $f^{(n)}(x) = (-2)^n e^{-2x}$, ...

$f(3) = e^{-6}$, $f'(3) = -2e^{-6}$, $f''(3) = (-2)^2 e^{-6}$, ..., $f^{(n)}(3) = (-2)^n e^{-6}$, ...

The required Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{(-2)^n e^{-6}}{n!}(x-3)^n$$
$$= e^{-6}\left[1 - 2(x-3) + \frac{(-2)^2}{2!}(x-3)^2 + \cdots + \frac{(-1)^n 2^n}{n!}(x-3)^n + \cdots\right]$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}e^{-6}(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-2)^n e^{-6}(x-3)^n}\right| = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)|x-3| = 0, \text{ so } R = \infty.$$

7. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, ...

$f\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $f'\left(-\frac{\pi}{6}\right) = \frac{1}{2}$, $f''\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $f'''\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, $f^{(4)}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, ...

The required Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}\left(-\frac{\pi}{6}\right)}{n!}\left(x+\frac{\pi}{6}\right)^n = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x+\frac{\pi}{6}\right) - \frac{1}{2!}\frac{\sqrt{3}}{2}\left(x+\frac{\pi}{6}\right)^2 - \frac{1}{3!}\frac{1}{2}\left(x+\frac{\pi}{6}\right)^3 + \frac{1}{4!}\frac{\sqrt{3}}{2}\left(x+\frac{\pi}{6}\right)^4 - \cdots$$
$$= \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x+\frac{\pi}{6}\right)^{2n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x+\frac{\pi}{6}\right)^{2n+1}$$

First consider the series with even powers:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\left(x+\frac{\pi}{6}\right)^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n\left(x+\frac{\pi}{6}\right)^{2n}}\right| = \lim_{n\to\infty}\frac{\left(x+\frac{\pi}{6}\right)^2}{(2n+1)(2n+2)} = 0, \text{ so } R = \infty.$$ We can similarly show that $R = \infty$ for the second series, so $R = \infty$ for the Taylor series as a whole.

14. $$f(x) = \frac{1}{1+3x} = \frac{1}{1+3(x-2)+6} = \frac{1}{7+3(x-2)} = \frac{1}{7}\cdot\frac{1}{1-\left[-\frac{3}{7}(x-2)\right]} = \frac{1}{7}\sum_{n=0}^{\infty}\left[-\frac{3}{7}(x-2)\right]^n$$
$$= \frac{1}{7}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{7}\right)^n(x-2)^n$$

The series converges for $\frac{3}{7}|x-2| < 1 \Leftrightarrow |x-2| < \frac{7}{3}$, so $R = \frac{7}{3}$.

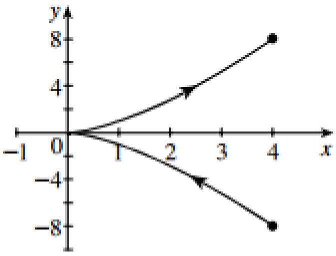
33. $$f(x) = (1-x)^{3/5} = [1+(-x)]^{3/5} = 1 + \frac{3}{5}(-x) + \frac{\frac{3}{5}\left(\frac{3}{5}-1\right)}{2!}x^2 + \cdots + \frac{\frac{3}{5}\left(\frac{3}{5}-1\right)\left(\frac{3}{5}-2\right)\cdots\left(\frac{3}{5}-n+1\right)}{n!}x^n + \cdots$$
$$= 1 - \frac{3}{5}x - 3\sum_{n=2}^{\infty}\frac{2\cdot 7\cdot 12\cdot\cdots\cdot(5n-8)}{n!5^n}x^n \text{ with } R = 1.$$

57. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. Because $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, we see that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \ln 2$.

10.2

7. a. $\left.\begin{array}{l} x = t^2 \\ y = t^3 \end{array}\right\} \Rightarrow x = y^{2/3},\ -8 \le y \le 8$

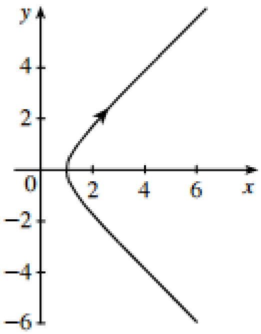
b.



Observe that y increases as t increases.

17. a. $\left.\begin{array}{l} x = \sec\theta \\ y = \tan\theta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x^2 = \sec^2\theta \\ y^2 = \tan^2\theta \end{array}\right\}$ From the identity $\sec^2\theta = 1 + \tan^2\theta$, we obtain $x^2 - y^2 = 1,\ x \ge 1$.

b.



10.3

4. $x = e^{2t}$, $y = \ln t \Rightarrow \frac{dx}{dt} = 2e^{2t}$ and $\frac{dy}{dt} = 1/t$. The slope of the tangent line at $t = 1$ is

$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{1/t}{2e^{2t}}\right|_{t=1} = \frac{1}{2e^2}.$$

17. $x = 3t^2 + 1$, $y = 2t^3 \Rightarrow \dfrac{dx}{dt} = 6t$ and $\dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{6t^2}{6t} = t$ and $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{1}{6t}$.

28. $x = \displaystyle\int_1^t \frac{\sin u}{u}\,du$, $y = \displaystyle\int_2^{\ln t} e^u\,du$. Using the Fundamental Theorem of Calculus, Part I, we have

$$\frac{dx}{dt} = \frac{d}{dt}\int_1^t \frac{\sin u}{u}\,du = \frac{\sin t}{t} \text{ and } \frac{dy}{dt} = \frac{d}{dt}\int_2^{\ln t} e^u\,du = e^{\ln t}\cdot\frac{d}{dt}\ln t = \frac{t}{t} = 1, \text{ so } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{\frac{\sin t}{t}} = \frac{t}{\sin t} \text{ and}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{1}{\frac{\sin t}{t}}\cdot\frac{d}{dt}\left(\frac{t}{\sin t}\right) = \frac{t}{\sin t}\cdot\frac{\sin t - t\cos t}{(\sin t)^2} = \frac{t\,(\sin t - t\cos t)}{\sin^3 t}.$$