3.1 Concept Questions

1. a. The absolute maximum value of a function f is the largest value assumed by $f(x)$ for all x in the domain of f. For example, if $f(x) = 1 - x^2$, then the absolute maximum value of f is 1.

b. f has a relative maximum value at a number c if there exists an open interval I containing c such that $f(c) \geq f(x)$ for all $x \in I$. For example, $f(x) = \sin x$ has relative maxima at $\frac{\pi}{2} \pm 2n\pi$, $n = 0, 1, 2, \ldots$.

2. a. A critical number of f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

b. f can have a relative extremum only at a critical number.

3. a. If f is continuous on a closed interval $[a, b]$, then f must attain an absolute maximum value and an absolute minimum value on $[a, b]$.

b. We find all critical numbers of f on (a, b), then evaluate f at these critical numbers as well as at the endpoints a and b. The largest and smallest numbers on the list are the absolute maximum and absolute minimum values of f on $[a, b]$, respectively.

3.1 Extrema of Functions

1. The absolute maximum value of f is 3, attained at $x = 1$. There is no absolute minimum value since f is not bounded below.

2. There is no absolute maximum value because f is not bounded above. The absolute minimum value is 0, attained at $x = 0$.

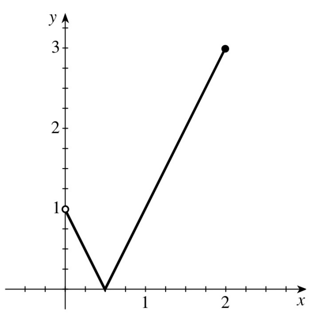
3. The absolute maximum value of f is 1, attained at $x = 2n$, n an integer. The absolute minimum value of f is 0, attained at $x = 2n + 1$, n an integer.

4. There is no absolute maximum value; $f(x)$ can be made as close to 2 as we please by taking x sufficiently close to $x = -2$, but the value 2 is never attained. The absolute minimum value of f is -2, attained at all values of $x > 0$.

5. The absolute maximum value of f is 37, attained at $x = 1$. The absolute minimum value of f is -5, attained at $x = 5$.

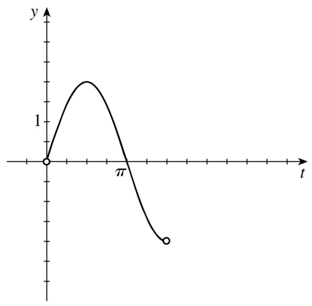
6. There is no absolute maximum value; $f(x)$ can be made as close to 1 as we please by taking x sufficiently large, but this value is never attained. There is no absolute minimum value because f is not bounded below.

18.



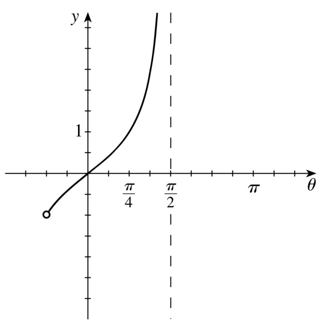
The absolute minimum value of g is 0, attained at $x = \frac{1}{2}$. The absolute maximum value of g is 3, attained at $x = 2$.

19.



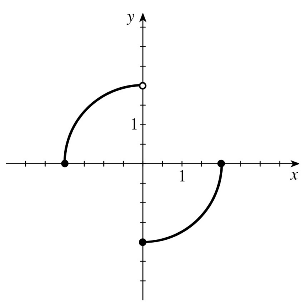
The absolute maximum value of f is 2, attained at $t = \frac{\pi}{2}$.

21.



f has no absolute extremum.

24.



The absolute minimum value of f is -2, attained at $x = 0$.

48. $f'(x) = \dfrac{d}{dx}\left(2x^4 - \frac{8}{3}x^3 - 8x^2 + 12\right) = 8x^3 - 8x^2 - 16x = 8x\left(x^2 - x - 2\right) = 8x(x-2)(x+1)$, so f has critical numbers -1, 0, and 2 on the interval $(-2, 3)$. We calculate $f(-2) = \frac{100}{3}$, $f(-1) = \frac{26}{3}$, $f(0) = 12$, $f(2) = -\frac{28}{3}$, and $f(3) = 30$, so f has an absolute maximum value of $\frac{100}{3}$ attained at $x = -2$ and an absolute minimum value of $-\frac{28}{3}$ attained at $x = 2$.

52. $f(x) = 2x + \dfrac{1}{x}$ is discontinuous at $x = 0$, which lies on the interval $(-1, 3)$. Since $\lim_{x\to 0^-}\left(2x + \dfrac{1}{x}\right) = -\infty$ and $\lim_{x\to 0^+}\left(2x + \dfrac{1}{x}\right) = \infty$, we see that f has no absolute minimum or absolute maximum value.

56. $g'(x) = \dfrac{d}{dx}\left[x\left(4-x^2\right)^{1/2}\right] = \left(4-x^2\right)^{1/2} + x\left(\tfrac{1}{2}\right)\left(4-x^2\right)^{-1/2}(-2x) = \left(4-x^2\right)^{-1/2}\left[\left(4-x^2\right)-x^2\right]$

$= \dfrac{2\left(2-x^2\right)}{\sqrt{4-x^2}}$ is discontinuous at $x = \pm 2$ and has zeros at $x = \pm\sqrt{2}$.

Therefore, g has the critical number $\sqrt{2}$ on the interval $(0, 2)$. We calculate $g(0) = 0$, $g\left(\sqrt{2}\right) = 2$, and $g(2) = 0$, so g has an absolute minimum value of 0 attained at $x = 0$ and at $x = 2$, and an absolute maximum value of 2 attained at $x = \sqrt{2}$.

58. $g'(x) = \dfrac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x$ is continuous everywhere and has zeros where $-\sin x - \cos x = 0 \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$ on the interval $(0, 2\pi)$. $g(0) = 1$, $g\left(\frac{3\pi}{4}\right) = -\sqrt{2}$, $g\left(\frac{7\pi}{4}\right) = \sqrt{2}$, and $g(2\pi) = 1$, so g has an absolute minimum value of $-\sqrt{2}$ attained at $x = \frac{3\pi}{4}$, and an absolute maximum value of $\sqrt{2}$ attained at $x = \frac{7\pi}{4}$.

60. $f'(x) = \dfrac{d}{dx}(x - \sin x) = 1 - \cos x$ is continuous everywhere and has zeros where $1 - \cos x = 0 \Leftrightarrow \cos x = 1 \Rightarrow x = 2n\pi$, n an integer. Since none of these lies on $(0, 2\pi)$, f has no critical number in that interval. $f(0) = 0$ and $f(2\pi) = 2\pi$, so f has an absolute minimum value of 0 attained at $x = 0$ and an absolute maximum value of 2π attained at $x = 2\pi$.

99. False. Consider $f(x) = x^3$. Here $f'(x) = 3x^2 = 0 \Leftrightarrow x = 0$, but f has neither a relative maximum nor a relative minimum at 0.

100. False. Consider $f(x) = x^{2/3}$. Then $f'(x) = \dfrac{2}{3x^{1/3}}$ is not defined at $x = 0$, but f has a relative minimum at 0.

101. False. Consider $f(x) = \begin{cases} -x & \text{if } -1 \le x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ x & \text{if } 0 < x \le 1 \end{cases}$ f is defined on the closed interval $[-1, 1]$, but does not have an absolute minimum value on $[-1, 1]$.

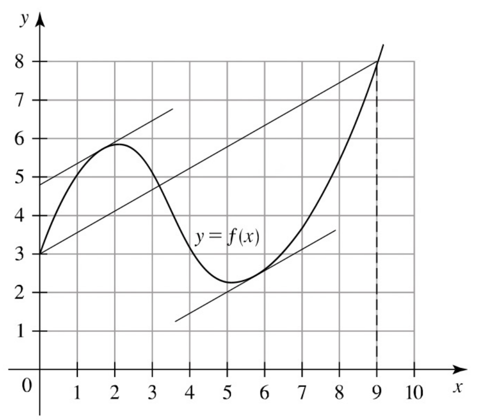
102. False. Consider $f(x) = x$ on the open interval $(0, 1)$. f is continuous on $(0, 1)$, but it has neither an absolute maximum nor an absolute minimum value on $(0, 1)$.

3.2 Concept Questions

1. See Theorem 1 on page 258. If f is constant on $[a, b]$, then its graph is a horizontal line segment and the slope of the tangent line to the graph of f is 0 at each number $c \in (a, b)$. If f is not a constant function, then its graph must rise above (or fall below) $f(a)$. Since f assumes the same value at b as it does at a, the graph must eventually turn down (or up), and consequently there is at least one turning point where the slope of the tangent line is 0.

2. See Theorem 2 on page 260. For a geometric interpretation of the Mean Value Theorem, see page 260.

3. **a.**



b. We find that $c_1 \approx 1.7$ and $c_2 \approx 5.4$.

24. $g(x) = x^4 - 2x^2 + x$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Furthermore, $g(0) = 0 = g(1)$. Therefore, by Rolle's Theorem, there exists at least one number c in $(0, 1)$ such that $g'(c) = 4c^3 - 4c + 1 = 0$. But $g'(x) = f(x)$, and so $f(c) = 0$, showing that f has at least one zero in $(0, 1)$.

28. Let $f(x) = \sin x$. If $a = b$, then $|f(a) - f(b)| = |\sin a - \sin b| = 0$ and $|a - b| = 0$. So $|\sin a - \sin b| = |a - b|$. Next, we assume that $a < b$. The function f is continuous on $[a, b]$ and differentiable on (a, b). Using the Mean Value Theorem, we see that there exists a number c in (a, b) such that $\dfrac{f(b) - f(a)}{b - a} = \dfrac{\sin b - \sin a}{b - a} = f'(c) = \cos c$. Thus, $\sin b - \sin a = (b - a)\cos c \Rightarrow |\sin b - \sin a| = |b - a|\,|\cos c|$. But $|\cos c| \le 1$ and so $|\sin b - \sin a| \le |b - a|$ and, combined with the previous result, we see that the inequality holds for all real numbers a and b.

37. Suppose f has two distinct fixed points c_1 and c_2 in I, with $c_1 < c_2$. Applying the Mean Value Theorem to f on the interval $[c_1, c_2]$ contained in I, we see that there exists a number c in (c_1, c_2) such that $f(c) = \dfrac{f(c_2) - f(c_1)}{c_2 - c_1} = \dfrac{c_2 - c_1}{c_2 - c_1} = 1$, contradicting the assumption that $f'(x) \ne 1$ for all x in I. This shows that f can have at most one fixed point in I.

45. False. Consider $f(x) = x^2$ on $[-1, 2]$. f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. Furthermore, $f'(0) = 2x|_{x=0} = 0$, but $f(-1) = 1 \ne 4 = f(2)$.

46. False. Consider f defined on $[-1, 2]$ by $f(x) = \begin{cases} -x & \text{if } -1 \le x < 0 \\ x^2 & \text{if } 0 \le x \le 2 \end{cases}$ Then f' is not differentiable at 0, and therefore fails to be differentiable on $(-1, 2)$. But $c = \frac{1}{2}$ on $(-1, 2)$ satisfies $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{4 - 1}{3} = 1$, which is equal to $f'\left(\frac{1}{2}\right) = 2x|_{x=1/2} = 1$.

47. True. If a and b are any two real numbers with $a < b$, then $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for some c in (a, b). But $f'(c) = 0$ everywhere. So $f(b) = f(a)$. Since a and b are arbitrary, f is a constant function.

48. True. By the Mean Value Theorem, if $x_1 \ne x_2$, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$, where $x_1 < c < x_2$. So $|f(x_2) - f(x_1)| = |f'(c)|\,|x_2 - x_1| \le |x_2 - x_1|$, since $|f'(c)| \le 1$.

49. True. $\dfrac{f(5) - f(2)}{5 - 2} = f'(c)$ for $2 < c < 5 \Rightarrow |f(5) - f(2)| = 3\,|f'(c)| \Leftrightarrow |f'(c)| = \dfrac{|f(5) - f(2)|}{3} \le \dfrac{6}{3} = 2$, contradicting $|f'(x)| > 2$ on $(2, 5)$.

50. True. By the Mean Value Theorem, there exists a number c with $1 < c < 3$ such that $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{f(3) - f(1)}{2} = f'(c) \Rightarrow f(3) - f(1) = 2f'(c)$. Since $2 \le f'(x) \le 4$ on $(1, 3)$, we have $4 \le f(3) - f(1) \le 8$ on $[1, 3]$.