4.5 Concept Questions

1. See page 405.
2. See page 406.
3. See pages 408 and 409.

4.5 The Fundamental Theorem of Calculus

1. a. $F'(x) = \dfrac{d}{dx} \int_2^x t^2\,dt = x^2$

b. $\int_2^x t^2\,dt = \frac{1}{3}t^3\Big|_2^x = \frac{1}{3}x^3 - \frac{8}{3}$

c. $\dfrac{d}{dx} f(x) = \dfrac{d}{dx}\left(\frac{1}{3}x^3 - \frac{8}{3}\right) = x^2$. The result is the same as that obtained in part a.

2. a. $G'(x) = \dfrac{d}{dx} \int_0^x \sqrt{3t+1}\,dt = \sqrt{3x+1}$

b. $\int_0^x \sqrt{3t+1}\,dt = \int_0^x (3t+1)^{1/2}\,dt = \frac{1}{3}\cdot\frac{2}{3}\cdot(3t+1)^{3/2}\Big|_0^x = \frac{2}{9}(3x+1)^{3/2} - \frac{2}{9}$

4. $G'(x) = \dfrac{d}{dx} \int_{-1}^x t\sqrt{t^2+1}\,dt = x\sqrt{x^2+1}$

6. $h'(x) = \dfrac{d}{dx}\displaystyle\int_x^3 \frac{t}{\sqrt{t+1}}\,dt = -\frac{d}{dx}\int_3^x \frac{t}{\sqrt{t+1}}\,dt = -\frac{x}{\sqrt{x+1}}$

8. $G'(x) = \dfrac{d}{dx} \int_0^{x^2} t\sin t\,dt = \left(x^2\sin x^2\right)\dfrac{d}{dx}\left(x^2\right) = 2x^3\sin x^2$

10. $h'(x) = \dfrac{d}{dx} \int_0^{x^2} \sin t^2\,dt = \sin\left(x^2\right)^2 \cdot \dfrac{d}{dx}\left(x^2\right) = 2x\sin x^4$

12. $G'(x) = \dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^5 \frac{\sin t^2}{t}\,dt = -\frac{d}{dx}\int_5^{x^{1/2}} \frac{\sin t^2}{t}\,dt = -\frac{\sin x}{\sqrt{x}}\cdot\frac{d}{dx}\left(x^{1/2}\right) = -\frac{\sin x}{2x}$

20. $\displaystyle\int_1^2 \frac{3x^4 - 2x^2 + 1}{2x^2}\,dx = \frac{1}{2}\int_1^2 \left(3x^2 - 2 + x^{-2}\right)dx = \frac{1}{2}\left(x^3 - 2x - \frac{1}{x}\right)\Bigg|_1^2 = \frac{1}{2}\left[\left(8 - 4 - \frac{1}{2}\right) - (1 - 2 - 1)\right] = \frac{11}{4}$

24. $\int_0^{\pi/2} (\sin x + 1)\,dx = -\cos x + x|_0^{\pi/2} = \left(-\cos\frac{\pi}{2} + \frac{\pi}{2}\right) - (-\cos 0 + 0) = \frac{\pi}{2} + 1$

28. $\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx = [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = (1-0) - (0-1) = 2$

30. $\int_0^{\pi} \sqrt{\sin x - \sin^3 x}\,dx = \int_0^{\pi} \sqrt{\sin x\left(1 - \sin^2 x\right)}\,dx = \int_0^{\pi} \sqrt{\sin x\cos^2 x}\,dx = \int_0^{\pi} \sqrt{\sin x}\,|\cos x|\,dx$

$= \int_0^{\pi/2} (\sin x)^{1/2}\cos x\,dx - \int_{\pi/2}^{\pi} (\sin x)^{1/2}\cos x\,dx = \left[\frac{2}{3}(\sin x)^{3/2}\right]_0^{\pi/2} - \left[\frac{2}{3}(\sin x)^{3/2}\right]_{\pi/2}^{\pi}$

$= \frac{2}{3} - \frac{2}{3}(-1) = \frac{4}{3}$

32. $\int_{-\pi}^{\pi/2} f(x)\,dx = \int_{-\pi}^{0}\left(x^2+1\right)dx + \int_0^{\pi/2}\cos x\,dx = \left[\frac{1}{3}x^3 + x\right]_{-\pi}^{0} + [\sin x]_0^{\pi/2} = \left[0 - \left(-\frac{\pi^3}{3} - \pi\right)\right] + 1 = \frac{\pi^3}{3} + \pi + 1$

34. Let $u = t+1$, so $du = dt$, $t = 0 \Rightarrow u = 1$, and $t = 2 \Rightarrow u = 3$. Then

$\int_0^2 (t+1)^{0.2}\,dt = \int_1^3 u^{0.2}\,du = \frac{1}{1.2}u^{1.2}\Big|_1^3 = \frac{1}{1.2}\left(3^{1.2} - 1\right).$

36. Let $u = 2x - 1$, so $du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$, $x = 1 \Rightarrow u = 1$, and $x = 5 \Rightarrow u = 9$. Then

$\int_1^5 \sqrt{2x-1}\,dx = \frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^9 = \frac{1}{3}\left(9^{3/2} - 1\right) = \frac{26}{3}.$

38. Let $u = 2x^2 - 1$, so $du = 4x\,dx \Rightarrow x\,dx = \frac{1}{4}\,du$, $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 7$. Then

$$\int_1^2 \frac{x}{\sqrt{2x^2-1}}\,dx = \frac{1}{4}\int_1^7 \frac{du}{u^{1/2}} = \frac{1}{4}\int_1^7 u^{-1/2}\,du = \frac{1}{2}\sqrt{u}\Big|_1^7 = \frac{1}{2}\left(\sqrt{7} - 1\right).$$

40. Let $u = 2x$, so $du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$, $x = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}$, and $x = \frac{\pi}{2} \Rightarrow u = \pi$. Then

$\int_{\pi/4}^{\pi/2} \sin 2x\,dx = \frac{1}{2}\int_{\pi/2}^{\pi} \sin u\,du = -\frac{1}{2}\cos u\Big|_{\pi/2}^{\pi} = -\frac{1}{2}(-1-0) = \frac{1}{2}.$

42. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta \Rightarrow \sin\theta\,d\theta = -du$, $\theta = 0 \Rightarrow u = 1$, and $\theta = \frac{\pi}{2} \Rightarrow u = 0$. Then

$\int_0^{\pi/2} \sqrt{\cos\theta}\sin\theta\,d\theta = -\int_1^0 u^{1/2}\,du = \int_0^1 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}.$

52. $A = \int_0^1 \left(x^3 + x\right)dx = \frac{1}{4}x^4 + \frac{1}{2}x^2\Big|_0^1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

54. $A = \int_0^3 \left(2 + \sqrt{x+1}\right)dx = \int_0^3 2\,dx + \int_0^3 \sqrt{x+1}\,dx$. Let $u = x + 1$ in the second integral, so $du = dx$, $x = 0 \Rightarrow u = 1$, and $x = 3 \Rightarrow u = 4$. Then $A = 2x|_0^3 + \int_1^4 u^{1/2}\,du = 6 + \frac{2}{3}u^{3/2}\Big|_1^4 = 6 + \frac{2}{3}(8-1) = \frac{32}{3}$.

56. $A = \int_{-\pi/2}^{\pi} |\sin x|\,dx = -\int_{-\pi/2}^{0} \sin x\,dx + \int_0^{\pi} \sin x\,dx = [\cos x]_{-\pi/2}^{0} - [\cos x]_0^{\pi} = (1-0) - (-1-1) = 3$

60. $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n}\left(\frac{k}{n}\right)^{1/3} = \int_0^1 x^{1/3}\,dx = \frac{3}{4}x^{4/3}\Big|_0^1 = \frac{3}{4}$

62. $\lim\limits_{n\to\infty} \frac{\pi}{2n}\sum\limits_{k=1}^{n}\cos\left(\frac{k\pi}{2n}\right) = \int_0^{\pi/2}\cos x\,dx = \sin x|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$

88. $\frac{d}{dy}\left[\int_0^x \sqrt{3 + 2\cos t}\,dt + \int_0^y \sin t\,dt\right] = \frac{d}{dy}(0) = 0 \Rightarrow \sqrt{3 + 2\cos x}\,\frac{dx}{dy} + \sin y = 0 \Rightarrow \frac{dx}{dy} = -\frac{\sin y}{\sqrt{3 + 2\cos x}}$

92. Let $F(x) = \int_2^x \sqrt{5 + t^2}\,dt$. Then

$F'(2) = \lim\limits_{h\to 0}\frac{F(2+h) - F(2)}{h} = \lim\limits_{h\to 0}\frac{\int_0^{2+h}\sqrt{5+t^2}\,dt - \int_0^2\sqrt{5+t^2}\,dt}{h} = \lim\limits_{h\to 0}\frac{\int_2^{2+h}\sqrt{5+t^2}\,dt}{h}$. Next, using FTC1, we find $F'(x) = \frac{d}{dx}\int_2^x \sqrt{5+t^2}\,dt = \sqrt{5+x^2} \Rightarrow F'(2) = \sqrt{5+2^2} = 3$, so we have $\lim\limits_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{5+t^2}\,dt = 3$.

104. True. $f\left((-x)^2\right) = f\left(x^2\right) \Rightarrow f\left(x^2\right)$ is an even function. Thus, $\int_{-a}^{a} f\left(x^2\right)dx = 2\int_0^a f\left(x^2\right)dx$.

105. True. Since f is even, we have $f\left((-x)^3\right) = f\left(-x^3\right) = f\left(x^3\right) \Rightarrow f\left(x^3\right)$ is an even function. Thus, $\int_{-a}^{a} f\left(x^3\right)dx = 2\int_0^a f\left(x^3\right)dx$.

106. True. Since f is odd, we have $f\left((-x)^3\right) = f\left(-x^3\right) = -f\left(x^3\right) \Rightarrow f\left(x^3\right)$ is an odd function. Thus, $\int_{-a}^{a} f\left(x^3\right)dx = 0$.

107. True. If f is even and g is odd, then fg^2 is even. Thus, $\int_{-a}^{a} f(x)\left[g(x)\right]^2 dx = 2\int_0^a f(x)\left[g(x)\right]^2 dx$.

Chapter 4 Review

Concept Review

1. a. $F' = f$ **b.** $F(x) + C$

2. a. $c \int f(x)\,dx$ **b.** $\int f(x)\,dx \pm \int g(x)\,dx$

3. a. unknown **b.** function

4. $g'(x)\,dx$; $f(u)\,du$

5. a. $\int_a^b f(x)\,dx$ **b.** minus

6. a. $\dfrac{1}{b-a}\int_a^b f(x)\,dx$ **b.** height; area

7. a. $f(x)$ **b.** $F(b) - F(a)$; antiderivative **c.** $\int_a^b f'(x)\,dx$

8. $2\int_0^a f(x)\,dx$; 0

9. $f(c) = \dfrac{1}{b-a}\int_a^b f(x)\,dx$

5.1 Concept Questions

1. a. $A = \int_a^c |f(x) - g(x)|\,dx$, $A = \int_c^e |f(y) - g(y)|\,dy$

b. $A = \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx$, $A = \int_c^d [g(y) - f(y)]\,dy + \int_d^e [f(y) - g(y)]\,dy$

5.1 Areas Between Curves

1. $A = \int_{-2}^{2} \left[(x+1) - \left(-x^2 - 1\right)\right] dx = \int_{-2}^{2} \left(x^2 + x + 2\right) dx = 2\int_0^2 \left(x^2 + 2\right) dx = 2\left(\frac{1}{3}x^3 + 2x\right)\Big|_0^2 = \frac{40}{3}$

2. $A = \int_0^2 \left[(x+2) - \left(x^3 - 4\right)\right] dx = \int_0^2 \left(-x^3 + x + 6\right) dx = -\frac{1}{4}x^4 + \frac{1}{2}x^2 + 6x\Big|_0^2 = 10$

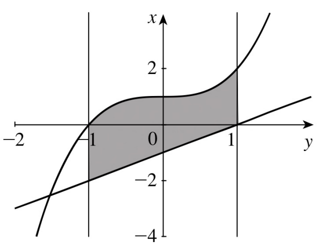
3. $A = \int_0^4 \left[\left(-x^2 + 4x\right) - \left(x - 2\sqrt{x}\right)\right] dx = \int_0^4 \left(-x^2 + 3x + 2x^{1/2}\right) dx = -\frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{4}{3}x^{3/2}\Big|_0^4 = \frac{40}{3}$

4. $A = \int_{-1}^{0} \left(x^2 - x^{1/3}\right) dx + \int_0^1 \left(x^{1/3} - x^2\right) dx = \left(\frac{1}{3}x^3 - \frac{3}{4}x^{4/3}\right)\Big|_{-1}^{0} + \left(\frac{3}{4}x^{4/3} - \frac{1}{3}x^3\right)\Big|_0^1$
$= -\left(-\frac{1}{3} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{1}{3}\right) = \frac{3}{2}$

5. $A = \int_0^1 \left[y^{1/3} - \left(2y^2 - 1\right)\right] dy = \int_0^1 \left(-2y^2 + y^{1/3} + 1\right) dy = -\frac{2}{3}y^3 + \frac{3}{4}y^{4/3} + y\Big|_0^1 = \frac{13}{12}$

6. $A = 2\int_0^1 \left[\left(y^4 - 2y^2 + 2\right) - \left(-y^{2/3}\right)\right] dy = 2\int_0^1 \left(y^4 - 2y^2 + y^{2/3} + 2\right) dy = 2\left(\frac{1}{5}y^5 - \frac{2}{3}y^3 + \frac{3}{5}y^{5/3} + 2y\right)\Big|_0^1 = \frac{64}{15}$

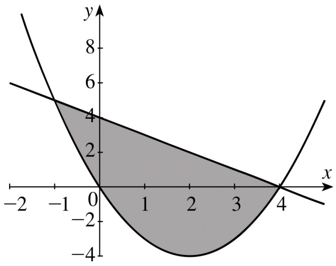
10. $A = \int_{-1}^{1} \left[\left(x^3 + 1\right) - (x - 1)\right] dx$
$= \int_{-1}^{1} \left(x^3 - x + 2\right) dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 + 2x\Big|_{-1}^{1} = 4$



12. Solve $x^2 - 4x = -x + 4 \Leftrightarrow x^2 - 3x - 4 = (x-4)(x+1) = 0$, giving $(-1, 5)$ and $(4, 0)$ as the points of intersection. Thus,

$$A = \int_{-1}^{4} \left[(-x+4) - \left(x^2 - 4x\right)\right] dx = \int_{-1}^{4} \left(-x^2 + 3x + 4\right) dx$$

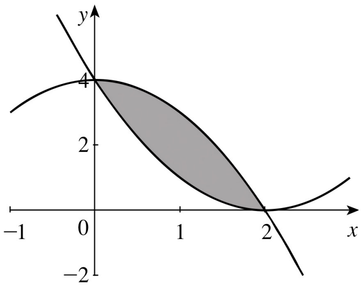
$$= -\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 4x \Big|_{-1}^{4} = \tfrac{125}{6}$$



14. Solve $(x-2)^2 = 4 - x^2 \Leftrightarrow x^2 - 4x + 4 = 4 - x^2 \Leftrightarrow$ $2x^2 - 4x = 2x(x-2) = 0$, giving $(0, 4)$ and $(2, 0)$ as the points of intersection. Thus,

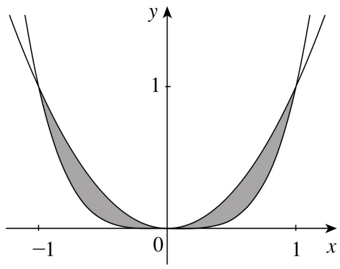
$$A = \int_{0}^{2} \left[\left(4 - x^2\right) - (x-2)^2\right] dx = \int_{0}^{2} \left(-2x^2 + 4x\right) dx$$

$$= -\tfrac{2}{3}x^3 + 2x^2 \Big|_{0}^{2} = \tfrac{8}{3}$$

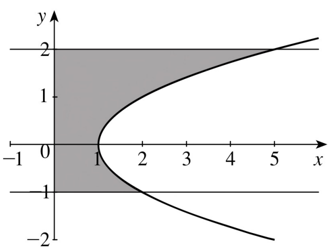


16. Solve $x^2 = x^4 \Leftrightarrow x^4 - x^2 = x^2(x-1)(x+1) = 0$, giving $(-1, 1)$, $(0, 0)$, and $(1, 1)$ as the points of intersection. By symmetry,

$$A = 2\int_{0}^{1} \left(x^2 - x^4\right) dx = 2\left(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_{0}^{1} = \tfrac{4}{15}.$$



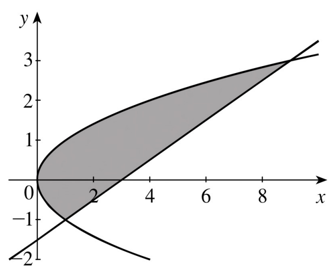
26. $A = \int_{-1}^{2} \left(y^2 + 1\right) dy = \tfrac{1}{3}y^3 + y \Big|_{-1}^{2} = 6$



28. Solve $y^2 = 2y + 3 \Leftrightarrow y^2 - 2y - 3 = (y-3)(y+1) = 0$, giving $(1, -1)$ and $(9, 3)$ as the points of intersection. Thus,

$$A = \int_{-1}^{3} \left[(2y+3) - y^2\right] dy = \int_{-1}^{3} \left(-y^2 + 2y + 3\right) dy$$

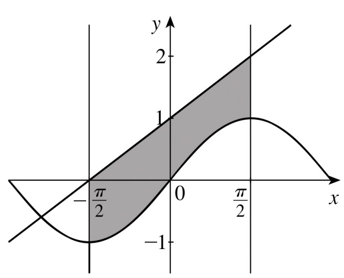
$$= -\tfrac{1}{3}y^3 + y^2 + 3y \Big|_{-1}^{3} = \tfrac{32}{3}$$



32. $A = \int_{-\pi/2}^{\pi/2} \left[\left(\tfrac{2}{\pi}x + 1\right) - \sin x\right] dx$

$$= 2\int_{0}^{\pi/2} 1\, dx = \pi$$

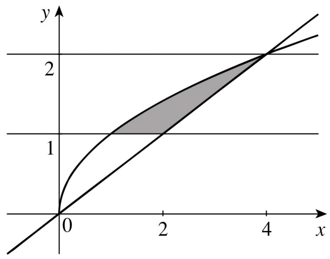
because $f(x) = \tfrac{2}{\pi}x - \sin x$ is odd.



44. **a.** $A = \int_1^2 \left(\sqrt{x} - 1\right) dx + \int_2^4 \left(\sqrt{x} - \frac{1}{2}x\right) dx$

$= \left(\frac{2}{3}x^{3/2} - x\right)\Big|_1^2 + \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right)\Big|_2^4 = \frac{2}{3}$

b. $A = \int_1^2 \left(2y - y^2\right) dy = \left(y^2 - \frac{1}{3}y^3\right)\Big|_1^2 = \frac{2}{3}$

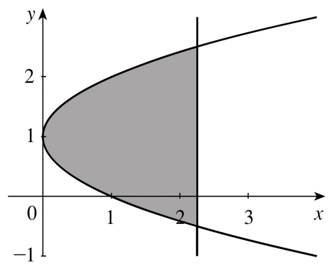


60. $x = (y-1)^2 \Leftrightarrow y = 1 \pm \sqrt{x}$, so

$A = \int_0^a \left[\left(1 + \sqrt{x}\right) - \left(1 - \sqrt{x}\right)\right] dx = \int_0^a 2x^{1/2}\, dx = \frac{4}{3}x^{3/2}\Big|_0^a$

$= \frac{4}{3}a^{3/2}$

We want $A = \frac{9}{2} \Rightarrow \frac{4}{3}a^{3/2} = \frac{9}{2} \Leftrightarrow a^{3/2} = \frac{27}{8} \Leftrightarrow a = \left(\frac{27}{8}\right)^{2/3} = \frac{9}{4}$.



69. False. Let $f(x) = 1$, $g(x) = x$, $a = 0$, and $b = 2$. Then $A = \int_a^b (1-x)\, dx = \left(x - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{2}$, so $A^2 = \frac{1}{4}$. But $\int_0^1 \left[f(x) - g(x)\right]^2 dx = \int_0^1 (1-x)^2\, dx = \int_0^1 \left(1 - 2x + x^2\right) dx = \left(x - x^2 + \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{3} \neq A^2$.

70. False. Let $f(x) = 2 - x$, $g(x) = \sqrt{x}$, $a = 0$, and $b = 2$. Then $\int_a^b \left[f(x) - g(x)\right] dx = \int_0^2 \left(2 - x - x^{1/2}\right) dx = \left(2x - \frac{1}{2}x^2 - \frac{2}{3}x^{3/2}\right)\Big|_0^2 = 2 - \frac{4\sqrt{2}}{3} \approx 0.1144 > 0$, but $f(x) < g(x)$ for $1 < x \le 2$.