

7.2 Trigonometric Integrals

$$\begin{aligned} 4. \int_0^{\pi/2} \sqrt{\cos x} \sin^3 x \, dx &= \int_0^{\pi/2} \cos^{1/2} x \cdot \sin^2 x \cdot \sin x \, dx = \int_0^{\pi/2} \cos^{1/2} x (1 - \cos^2 x) \sin x \, dx \\ &= \int_0^{\pi/2} (\cos^{1/2} x - \cos^{5/2} x) \sin x \, dx \end{aligned}$$

Let $u = \cos x$, so $du = -\sin x \, dx$, $x = 0 \Rightarrow u = 1$, and $x = \frac{\pi}{2} \Rightarrow u = 0$. Then

$$\int_0^{\pi/2} \sqrt{\cos x} \sin^3 x \, dx = -\int_1^0 (u^{1/2} - u^{5/2}) \, du = -\left. \left(\frac{2}{3} u^{3/2} - \frac{2}{7} u^{7/2} \right) \right|_1^0 = -\left(-\frac{2}{3} + \frac{2}{7} \right) = \frac{8}{21}.$$

$$7. \int_0^{\pi} \cos^2 \frac{x}{2} \, dx = \int_0^{\pi} \frac{1 + \cos x}{2} \, dx = \left. \frac{x + \sin x}{2} \right|_0^{\pi} = \frac{\pi}{2}$$

$$\begin{aligned} 22. \int \tan^5 x \sec^3 x \, dx &= \int \tan^4 x \sec^2 x \sec x \tan x \, dx = \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \, dx \\ &= \int (\sec^6 x - 2 \sec^4 x + \sec^2 x) \sec x \tan x \, dx = \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C \end{aligned}$$

$$24. \int_0^{\pi/4} \sec^2 x \tan^2 x \, dx = \left. \frac{1}{3} \tan^3 x \right|_0^{\pi/4} = \frac{1}{3}$$

$$\begin{aligned} 39. \int_0^{\pi/2} \sin x \cos 2x \, dx &= \frac{1}{2} \int_0^{\pi/2} [\sin(1-2)x + \sin(1+2)x] \, dx = \frac{1}{2} \int_0^{\pi/2} (-\sin x + \sin 3x) \, dx \\ &= \left. \frac{1}{2} \left(\cos x - \frac{1}{3} \cos 3x \right) \right|_0^{\pi/2} = \frac{1}{2} \left(0 - \frac{2}{3} \right) = -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} 42. \int \frac{\sin^3 x}{\sec^2 x} \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx = \int (\cos^2 x - \cos^4 x) \sin x \, dx \\ &= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C \end{aligned}$$

$$\begin{aligned} 44. \int_0^{\pi/2} \frac{\sin t}{1 + \cos t} \, dt. & \text{ Let } u = 1 + \cos t, \text{ so } du = -\sin t \, dt, t = 0 \Rightarrow u = 2, \text{ and } t = \frac{\pi}{2} \Rightarrow u = 1. \text{ Then} \\ \int_0^{\pi/2} \frac{\sin t}{1 + \cos t} \, dt &= -\int_2^1 \frac{du}{u} = -\ln |u| \Big|_2^1 = -\ln 1 + \ln 2 = \ln 2. \end{aligned}$$

$$47. \int \frac{1 - \tan^2 x}{\sec^2 x} \, dx = \int \frac{1 - (\sec^2 x - 1)}{\sec^2 x} \, dx = \int \frac{2 - \sec^2 x}{\sec^2 x} \, dx = \int (2 \cos^2 x - 1) \, dx = \int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$$

7.3 Trigonometric Substitutions

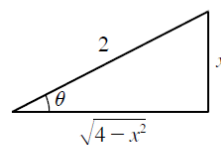
3. $\int x\sqrt{4-x^2} dx$. Let $x = 2 \sin \theta$, so $dx = 2 \cos \theta d\theta$ and

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = 2 \cos \theta. \text{ Then}$$

$$\int x\sqrt{4-x^2} dx = \int (2 \sin \theta) (2 \cos \theta) (2 \cos \theta) d\theta = 8 \int \cos^2 \theta \sin \theta d\theta$$

$$= -\frac{8 \cos^3 \theta}{3} + C = -\frac{8}{3} \left(\frac{\sqrt{4-x^2}}{2} \right)^3 + C = -\frac{(4-x^2)^{3/2}}{3} + C$$

Note that the problem can be solved more easily using the substitution $u = 4 - x^2$.

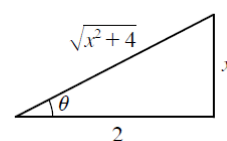


7. $\int \frac{dx}{x^2\sqrt{x^2+4}}$. Let $x = 2 \tan \theta$, so $dx = 2 \sec^2 \theta d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4 \tan^2 \theta + 4} = 2\sqrt{\tan^2 \theta + 1} = 2 \sec \theta. \text{ Then}$$

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2 \sec^2 \theta d\theta}{(4 \tan^2 \theta) (2 \sec \theta)} = \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{4} \int (\sin \theta)^{-2} \cos \theta d\theta$$

$$= -\frac{1}{4 \sin \theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$



22. $I = \int_2^4 \frac{\sqrt{x^2-4}}{x^4} dx$. Let $x = 2 \sec \theta$, so $dx = 2 \sec \theta \tan \theta d\theta$,

$$\sqrt{x^2-4} = \sqrt{4 \sec^2 \theta - 4} = 2 \tan \theta, \quad x = 2 \Rightarrow \theta = 0, \text{ and } x = 4 \Rightarrow \theta = \frac{\pi}{3}. \text{ Then}$$

$$I = \int_2^4 \frac{\sqrt{x^2-4}}{x^4} dx = \int_0^{\pi/3} \frac{2 \tan \theta (2 \sec \theta \tan \theta d\theta)}{(2 \sec \theta)^4} = \frac{1}{4} \int_0^{\pi/3} \sin^2 \theta \cos \theta d\theta = \frac{1}{12} \sin^3 \theta \Big|_0^{\pi/3} = \frac{1}{12} \left(\frac{\sqrt{3}}{2} \right)^3 = \frac{\sqrt{3}}{32}.$$

25. $I = \int e^x \sqrt{4-e^{2x}} dx$. Let $u = e^x$, so $du = e^x dx$. Then $I = \int \sqrt{4-u^2} du$. Next, let $u = 2 \sin \theta$, so $du = 2 \cos \theta d\theta$ and

$$\sqrt{4-u^2} = \sqrt{4-4\sin^2 \theta} = 2\sqrt{1-\sin^2 \theta} = 2 \cos \theta. \text{ Then}$$

$$I = \int (2 \cos \theta) (2 \cos \theta d\theta) = 4 \int \cos^2 \theta d\theta = 4 \int \frac{1+\cos 2\theta}{2} d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2 \sin \theta \cos \theta + C$$

$$= 2 \left[\sin^{-1} \left(\frac{1}{2} u \right) + \frac{1}{2} u \left(\frac{1}{2} \sqrt{4-u^2} \right) \right] + C$$

$$\text{Therefore, } \int e^x \sqrt{4-e^{2x}} dx = 2 \sin^{-1} \left(\frac{1}{2} e^x \right) + \frac{1}{2} e^x \sqrt{4-e^{2x}} + C.$$

26. $I = \int e^t \sqrt{1+e^{2t}} dt$. Let $u = e^t$, so $du = e^t dt$. Then $I = \int \sqrt{1+u^2} du$. Next, let

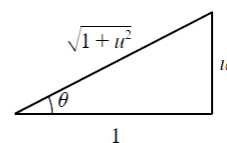
$$u = \tan \theta, \text{ so } du = \sec^2 \theta d\theta \text{ and } \sqrt{1+u^2} = \sqrt{1+\tan^2 \theta} = \sec \theta. \text{ Then}$$

$$I = \int \sec \theta \sec^2 \theta d\theta = \int \sec^3 \theta d\theta$$

$$= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C \text{ (see Example 5 in Section 7.1)}$$

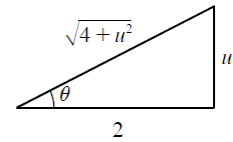
$$= \frac{1}{2} \sqrt{1+u^2} u + \frac{1}{2} \ln \left| \sqrt{1+u^2} + u \right| + C$$

$$\text{Thus, } \int e^t \sqrt{1+e^{2t}} dt = \frac{1}{2} \left[e^t \sqrt{1+e^{2t}} + \ln \left(\sqrt{1+e^{2t}} + e^t \right) \right] + C.$$



31. $I = \int \frac{dx}{(x^2 + 4x + 8)^2}$. Observe that

$$x^2 + 4x + 8 = [x^2 + 4x + 2^2] + 8 - 4 = 4 + (x + 2)^2, \text{ so } I = \int \frac{dx}{[4 + (x + 2)^2]^2}. \text{ Let}$$



$$u = x + 2, \text{ so } du = dx. \text{ Then } I = \int \frac{du}{(4 + u^2)^2}. \text{ Next, let } u = 2 \tan \theta, \text{ so } du = 2 \sec^2 \theta d\theta$$

and $4 + u^2 = 4 + 4 \tan^2 \theta = 4 \sec^2 \theta$. Then

$$\begin{aligned} I &= \int \frac{2 \sec^2 \theta d\theta}{(4 \sec^2 \theta)^2} = \frac{1}{8} \int \frac{d\theta}{\sec^2 \theta} = \frac{1}{8} \int \cos^2 \theta d\theta = \frac{1}{8} \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{16} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{1}{16} \theta + \frac{1}{16} \sin \theta \cos \theta + C = \frac{1}{16} \tan^{-1} \left(\frac{u}{2} \right) + \frac{1}{16} \frac{u}{\sqrt{4 + u^2}} \frac{2}{\sqrt{4 + u^2}} + C = \frac{1}{16} \tan^{-1} \left(\frac{u}{2} \right) + \frac{u}{8(4 + u^2)} + C \end{aligned}$$

$$\text{Therefore, } I = \frac{1}{16} \tan^{-1} \left(\frac{x + 2}{2} \right) + \frac{x + 2}{8(x^2 + 4x + 8)} + C.$$