13. 3

14. $f_x(x, y) = \frac{\partial}{\partial x}\left(e^x \cos y + e^y \sin x\right) = e^x \cos y + e^y \cos x$ and $f_y(x, y) = \frac{\partial}{\partial y}\left(e^x \cos y + e^y \sin x\right) = -e^x \sin y + e^y \sin x$.

30. $\frac{\partial}{\partial x}\left(x^2 y + xz + yz^2\right) = \frac{\partial}{\partial x}(8) \Rightarrow 2xy + z + x\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2xy + z}{x + 2yz}$ and
$\frac{\partial}{\partial y}\left(x^2 y + xz + yz^2\right) = \frac{\partial}{\partial y}(8) \Rightarrow x^2 + x\frac{\partial z}{\partial y} + z^2 + 2yz\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{x^2 + z^2}{x + 2yz}$.

37. $\frac{\partial w}{\partial u} = \frac{\partial}{\partial u}\left[\cos(2u - v) + \sin(2u + v)\right] = -2\sin(2u - v) + 2\cos(2u + v)$,
$\frac{\partial w}{\partial v} = \frac{\partial}{\partial v}\left[\cos(2u - v) + \sin(2u + v)\right] = \sin(2u - v) + \cos(2u + v)$, $\frac{\partial^2 w}{\partial u^2} = -4\cos(2u - v) - 4\sin(2u + v)$,
$\frac{\partial^2 w}{\partial v^2} = -\cos(2u - v) - \sin(2u + v)$, and $\frac{\partial^2 w}{\partial u\,\partial v} = \frac{\partial^2 w}{\partial v\,\partial u} = 2\cos(2u - v) - 2\sin(2u + v)$.

43. $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x\cos y + y\sin x) = \cos y + y\cos x \Rightarrow \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}(\cos y + y\cos x) = -\sin y + \cos x \Rightarrow$
$\frac{\partial^3 z}{\partial x\,\partial y\,\partial x} = \frac{\partial}{\partial x}(-\sin y + \cos x) = -\sin x$

47. $f_x(x, y) = \frac{\partial}{\partial x}\left(x\sin^2 y + y^2\cos x\right) = \sin^2 y - y^2\sin x \Rightarrow f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\sin^2 y - y^2\sin x\right) = 2\sin y\cos y - 2y\sin x$
and $f_y(x, y) = \frac{\partial}{\partial y}\left(x\sin^2 y + y^2\cos x\right) = 2x\sin y\cos y + 2y\cos x \Rightarrow$
$f_{yx}(x, y) = \frac{\partial}{\partial x}(2x\sin y\cos y + 2y\cos x) = 2\sin y\cos y - 2y\sin x$, so $f_{xy} = f_{yx}$.

13.5

8. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = \sqrt{y^2+z^2}\left(-\dfrac{1}{t^2}\right) + \dfrac{xy}{\sqrt{y^2+z^2}}\left[e^{-t}\left(-\cos t - \sin t\right)\right] + \dfrac{xz}{\sqrt{y^2+z^2}}\left[e^{-t}\left(-\sin t + \cos t\right)\right]$. Observe that $\sqrt{y^2+z^2} = e^{-t}$, so we can write

$$\frac{dw}{dt} = -\frac{e^{-t}}{t^2} - xy\left(\cos t + \sin t\right) + xz\left(-\sin t + \cos t\right) = -\frac{e^{-t}}{t^2} - x\left(y\cos t + y\sin t + z\sin t - z\cos t\right).$$

11. $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} = \left(e^x\cos y\right)\left(\dfrac{2u}{u^2+v^2}\right) + \left(-e^x\sin y\right)\left(\dfrac{1}{2}\dfrac{\sqrt{uv}}{u}\right) = e^x\left(\dfrac{2u\cos y}{u^2+v^2} - \dfrac{\sqrt{uv}\sin y}{2u}\right)$ and $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} = \left(e^x\cos y\right)\left(\dfrac{2v}{u^2+v^2}\right) + \left(-e^x\sin y\right)\left(\dfrac{1}{2}\dfrac{\sqrt{uv}}{v}\right) = e^x\left(\dfrac{2v\cos y}{u^2+v^2} - \dfrac{\sqrt{uv}\sin y}{2v}\right)$.

20. $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = \left(\sqrt{y} + \dfrac{1}{2\sqrt{x}}\right)(1) + \dfrac{x}{2\sqrt{y}}(-7)$. If $s = 4$ and $t = 1$, then $x = 9$ and $y = 9$, so $\dfrac{\partial z}{\partial t} = \left(\sqrt{9} + \dfrac{1}{2\sqrt{9}}\right)(1) + \dfrac{9}{2\sqrt{9}}(-7) = -\dfrac{22}{3}$.

27. Differentiating each equation in the system with respect to x, we obtain $\dfrac{\partial}{\partial x}(x) = \dfrac{\partial}{\partial x}\left(u^2+v^2\right) = 2u\dfrac{\partial u}{\partial x} + \partial v\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial}{\partial x}(y) = \dfrac{\partial}{\partial x}\left(u^2-v^2\right) = 2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x}$. Since y is an independent variable, $\dfrac{\partial}{\partial x}(y) = 0$, so we have $2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x} = 1$ and $2u\dfrac{\partial u}{\partial x} - 2v\dfrac{\partial v}{\partial x} = 0$. Using Cramer's Rule, we obtain $\dfrac{\partial u}{\partial x} = \dfrac{\begin{vmatrix} 1 & 2v \\ 0 & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 2v \\ 2u & -2v \end{vmatrix}} = \dfrac{-2v}{-8uv} = \dfrac{1}{4u}$ and

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} 2u & 1 \\ 2u & 0 \end{vmatrix}}{\begin{vmatrix} 2u & 2v \\ 2u & -2v \end{vmatrix}} = \frac{-2u}{-8uv} = \frac{1}{4v}.$$ Similarly, we obtain $\dfrac{\partial u}{\partial y} = \dfrac{1}{4u}$ and $\dfrac{\partial v}{\partial y} = -\dfrac{1}{4v}$.

33. Here $F(x,y,z) = x^2 + xy - x^2z + yz^2 = 0$, so $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2x+y-2xz}{-x^2+2yz} = \dfrac{2x+y-2xz}{x^2-2yz}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{x+z^2}{-x^2+2yz} = \dfrac{x+z^2}{x^2-2yz}$.