1.2 Concept Questions

1. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

- the **Sum Law** states that $\lim_{x\to a} \left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$.
- the **Product Law** states that $\lim_{x\to a} \left[f(x) \cdot g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$.
- the **Constant Multiple Law** states that $\lim_{x\to a} \left[cf(x)\right] = c \lim_{x\to a} f(x) = cL$ for every c.
- the **Quotient Law** states that $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, provided that $M \neq 0$.
- the **Root Law** states that $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$, provided that $L > 0$ if n is even.

2. a.
$$\begin{aligned}
\lim_{x\to 2} \left(3x^2 - 2x + 1\right) &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 2x + \lim_{x\to 2} 1 \quad \text{(Sum Law)}\\
&= 3 \lim_{x\to 2} x^2 - 2 \lim_{x\to 2} x + \lim_{x\to 2} 1 \quad \text{(Constant Multiple Law)}\\
&= 3\left[\left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right)\right] - 2 \lim_{x\to 2} x + \lim_{x\to 2} 1 \quad \text{(Product Law)}\\
&= 3 \cdot 2 \cdot 2 - 2 \cdot 2 + 1 \quad \text{(Law 2 and Law 1)}\\
&= 9
\end{aligned}$$

b.
$$\begin{aligned}
\lim_{x\to 3} \frac{x^2+4}{2x+3} &= \frac{\lim_{x\to 3} \left(x^2+4\right)}{\lim_{x\to 3} (2x+3)} \quad \text{(Quotient Law)}\\
&= \frac{\lim_{x\to 3} x^2 + \lim_{x\to 3} 4}{\lim_{x\to 3} 2x + \lim_{x\to 3} 3} \quad \text{(Sum Law)}\\
&= \frac{\left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) + \lim_{x\to 3} 4}{2 \lim_{x\to 3} x + \lim_{x\to 3} 3} \quad \text{(Product Law and Constant Multiple Law)}\\
&= \frac{3 \cdot 3 + 4}{2 \cdot 3 + 3} \quad \text{(Law 2 and Law 1)}\\
&= \frac{13}{9}
\end{aligned}$$

3. a.
$$\begin{aligned}
\lim_{x\to 4} \sqrt{x}\left(2x^2+1\right) &= \left(\lim_{x\to 4} \sqrt{x}\right)\left[\lim_{x\to 4} \left(2x^2+1\right)\right] \quad \text{(Product Law)}\\
&= \sqrt{\lim_{x\to 4} x}\left(\lim_{x\to 4} 2x^2 + \lim_{x\to 4} 1\right) \quad \text{(Root Law and Sum Law)}\\
&= \sqrt{\lim_{x\to 4} x}\left(2 \lim_{x\to 4} x^2 + \lim_{x\to 4} 1\right) \quad \text{(Constant Multiple Law)}\\
&= \sqrt{\lim_{x\to 4} x}\left[2\left(\lim_{x\to 4} x\right)\left(\lim_{x\to 4} x\right) + \lim_{x\to 4} 1\right] \quad \text{(Product Law)}\\
&= \sqrt{4}\,(2 \cdot 4 \cdot 4 + 1) \quad \text{(Law 2 and Law 1)}\\
&= 66
\end{aligned}$$

b. $$\lim_{x\to 1}\left(\frac{2x^2+x+5}{x^4+1}\right)^{3/2} = \left(\lim_{x\to 1}\frac{2x^2+x+5}{x^4+1}\right)^{3/2} \quad \text{(Root Law)}$$

$$= \left[\frac{\lim_{x\to 1}\left(2x^2+x+5\right)}{\lim_{x\to 1}\left(x^4+1\right)}\right]^{3/2} \quad \text{(Quotient Law)}$$

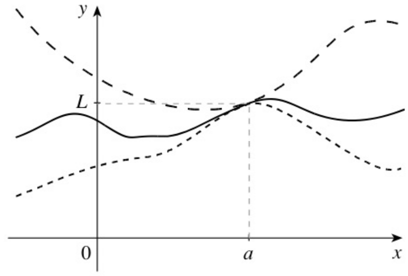
$$= \left[\frac{2\left(\lim_{x\to 1} x\right)^2 + \lim_{x\to 1} x + \lim_{x\to 1} 5}{\left(\lim_{x\to 1} x\right)^4 + \lim_{x\to 1} 1}\right]^{3/2} \quad \text{(Sum Law, Constant Multiple Law, and Product Law)}$$

$$= \left(\frac{2\cdot 1^2+1+5}{1^4+1}\right)^{3/2} \quad \text{(Law 2 and Law 1)}$$

$$= \left(\frac{8}{2}\right)^{3/2} = 8$$

4. If $f(x) \le g(x) \le h(x)$ for all x in an open interval containing a, except possibly at a, and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$.

If $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if both $f(x)$ and $h(x)$ approach L as x approaches a, then $g(x)$ must approach L as well.



16. $$\lim_{w\to 0}\frac{\sqrt{w+1}-\sqrt{w^2+4}}{(w+2)^2-(w+1)^2} = \frac{1-2}{4-1} = -\frac{1}{3}$$

19. $$\lim_{x\to \pi/4}\frac{\sin x}{x} = \frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}\cdot\frac{4}{\pi} = \frac{2\sqrt{2}}{\pi}$$

22. $$\lim_{x\to \pi/4}\frac{\tan^2 x}{1+\cos x} = \frac{\left(\tan\frac{\pi}{4}\right)^2}{1+\cos\frac{\pi}{4}} = \frac{1^2}{1+\frac{\sqrt{2}}{2}} = \frac{2}{2+\sqrt{2}} = 2-\sqrt{2}$$

37. Incorrect. $f(x) = \dfrac{x^2-9}{x+3} = \dfrac{(x+3)(x-3)}{x+3} = x-3$ *provided* $x \neq -3$. $\lim_{x\to -3} f(x)$ cannot be found by substituting $x = 3$ in $f(x)$.

38. Correct. Since the denominator of $\dfrac{x^2-9}{x+3}$ is 0 at $x=-3$, we cannot find the limit by direct substitution. However, $x-3$ is equivalent to $\dfrac{x^2-9}{x+3}$ provided that $x \neq -3$. Therefore, $\lim_{x\to -3}\dfrac{x^2-9}{x+3} = \lim_{x\to -3}\dfrac{(x+3)(x-3)}{x+3} = \lim_{x\to -3}(x-3) = -6$.

44. $$\lim_{x\to 2^+}\frac{x+1}{x-2} = \infty$$

46. $$\lim_{x\to 2}\frac{x^2-x-2}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

58. $\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}\cdot\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}=\lim_{h\to 0}\frac{a+h-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)}$

$=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{2\sqrt{a}}=\frac{\sqrt{a}}{2a}$

66. $\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\cdot\frac{2}{1}\right)=2\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=2\cdot 1=2$. We have made the substitution $\theta=2x$ at the third step.

68. $\lim_{x\to 0}\frac{\tan 2x}{3x}=\lim_{x\to 0}\left(\frac{\sin 2x}{\cos 2x}\cdot\frac{1}{3x}\right)=\left(\lim_{x\to 0}\frac{1}{\cos 2x}\right)\left[\lim_{x\to 0}\frac{\sin 2x}{\frac{3}{2}(2x)}\right]=1\cdot\frac{2}{3}\cdot\lim_{x\to 0}\frac{\sin 2x}{2x}=\frac{2}{3}\cdot 1=\frac{2}{3}$

86. $\lim_{x\to -2^-}f(x)=\lim_{x\to -2^-}\frac{x^3-16}{x}=12$ and $\lim_{x\to -2^+}f(x)=\lim_{x\to -2^+}\left(-x^2-4x+8\right)=12$. Since $\lim_{x\to -2^-}f(x)=\lim_{x\to -2^+}f(x)=12$, $\lim_{x\to -2}f(x)$ exists and has a value of 12.

88. $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\left(\sqrt{1-x}+2\right)=2$ and $\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}\left(1+x^{3/2}\right)=2$. Since $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)=2$, we conclude that $\lim_{x\to 1}f(x)$ exists and has a value of 2.

99. False. Neither $\lim_{x\to 2}\frac{3x}{x-2}$ nor $\lim_{x\to 2}\frac{2}{x-2}$ exists, and so the Sum and Difference Laws for limits do not apply.

100. True. In fact, using the Quotient Law, $\lim_{x\to 1}\frac{x^2+3x-4}{x^2-2x-3}=\frac{\lim_{x\to 1}\left(x^2+3x-4\right)}{\lim_{x\to 1}\left(x^2-2x-3\right)}=\frac{0}{-4}=0$.

101. False. Consider $f(x)=\frac{x}{x-1}$ and $g(x)=\frac{1}{x-1}$. Then $\lim_{x\to 1}\left[f(x)-g(x)\right]=\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{x-1}\right)=\lim_{x\to 1}1=1$, but neither of $\lim_{x\to 1}\frac{x}{x-1}$ and $\lim_{x\to 1}\frac{1}{x-1}$ exists.

102. False. Let $f(x)=-2$, $h(x)=2$, and $g(x)=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x\geq 0\end{cases}$ Then $f(x)\leq g(x)\leq h(x)$ for all x. Also, $\lim_{x\to 0}f(x)$ and $\lim_{x\to 0}h(x)$ both exist, but $\lim_{x\to 0}g(x)$ does not exist.

1.4 Concept Questions

1. a. If f is continuous at a, then $\lim_{x\to a}f(x)=f(a)$. For example, $f(x)=x+1$ is continuous at 0 since $\lim_{x\to 0}f(x)=f(0)=1$.

b. If f is continuous from the right at a, then $\lim_{x\to a^+}f(x)=f(a)$. For example, $f(x)=\sqrt{x+1}-2$ is continuous from the right at -1, since $\lim_{x\to -1^+}f(x)=f(-1)=-2$.

c. If f is continuous from the left at a, then $\lim_{x\to a^-}f(x)=f(a)$. For example, $f(x)=\sqrt{1-x}$ is continuous from the left at 1, since $\lim_{x\to 1^-}f(x)=f(1)=0$.

2. **a.** f is continuous on an open interval (a, b) if it is continuous at every number in the interval. For example, $f(x) = \tan x$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

b. f is continuous on a closed interval $[a, b]$ if it is continuous on (a, b) and is also continuous from the right at a and continuous from the left at b. For example, $f(x) = \sqrt{4 - x^2}$ is continuous on $[-2, 2]$.

4. **a.** f has a removable discontinuity at a if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ and f can be made continuous at a by defining or redefining $f(a)$.

b. If f is continuous from the left at a and from the right at a, then $\lim_{x \to a^-} f(x) = f(a)$ and $\lim_{x \to a^+} f(x) = f(a)$. Therefore, $\lim_{x \to a} f(x) = f(a)$ and so f is continuous at a.

1. f is continuous everywhere.

2. f is discontinuous at -1.

3. f is discontinuous at ± 1.

4. f is discontinuous at $0, \pm 1, \pm 2, \ldots$.

5. f is discontinuous at 0.

6. f is continuous everywhere. Note that the graph of f is squeezed between the graphs of $g(x) = -x$ and $h(x) = x$. Since $\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = 0$, it follows that $\lim_{x \to 0} f(x) = 0$ by the Squeeze Theorem.

12. The denominator of the function f is equal to 0 when $(x - 3)(x + 1) = 0$, that is, when $x = 3$ or $x = -1$. So f is discontinuous at 3 and -1.

16. The denominator of the quotient $\dfrac{x + 1}{|x + 1|}$ is equal to 0 when $x + 1 = 0$, or $x = -1$, so f is discontinuous at -1.

22. None. Since $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} \dfrac{(x + 3)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 3) = 5$ and $f(2) = 5$, f is continuous at 2. So f is continuous on $(-\infty, \infty)$.

24. Since $\lim_{x \to 0} (-|x| + 1) = 1$ and $f(0) = 0$, f is discontinuous at 0.

28. Since $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2} = \lim_{x \to -2} \dfrac{(x - 2)(x + 2)}{x + 2} = \lim_{x \to -2} (x - 2) = -4$, we define $f(-2) = k = -4$, that is, take $k = -4$.

30. We require that $\lim_{x \to 2^-} (kx + 1) = \lim_{x \to 2^+} \left(kx^2 - 3\right)$, or $2k + 1 = 4k - 3$. Solving this last equation, we obtain $k = 2$.

32. Since $\lim_{x \to 0^-} (x \cot kx) = \lim_{x \to 0^-} \left(\dfrac{x}{\sin kx} \cdot \dfrac{\cos kx}{1}\right) = \lim_{x \to 0^-} \left(\dfrac{1}{k} \cdot \dfrac{kx}{\sin kx}\right) \cdot \lim_{x \to 0^-} (\cos kx) = \dfrac{1}{k}$, we define $f(0) = 1/k$. Then $0 + c = 1/k$, or $c = 1/k$.

46. f is a quotient of two continuous functions, but 0 is not in its domain, and so f is continuous on $(-\infty, 0) \cup (0, \infty)$.

50. The absolute value function is continuous, so

$$\lim_{x\to-1}\left|\frac{x^2-x-2}{x+1}\right| = \left|\lim_{x\to-1}\frac{(x+1)(x-2)}{x+1}\right| = \left|\lim_{x\to-1}(x-2)\right| = |-3| = 3.$$

60. $f(x) = x^2 - 4x + 6$ is continuous on $[0,3]$. $f(0) = 6$ and $f(3) = 3$. Since $f(3) \le 3 \le f(0)$, there exists a number c in $[0,3]$ such that $f(c) = 3$. To find c we solve $x^2 - 4x + 6 = 3 \Rightarrow x^2 - 4x + 3 = (x-3)(x-1) = 0$ giving $x = 1$ or 3. Therefore $c = 1$ or 3.

64. $f(x) = x^4 - 2x^3 - 3x^2 + 7$ is continuous on $[1,2]$. $f(1) = 3 > 0$ and $f(2) = -5 < 0$. Therefore, by Theorem 7, $f(x) = 0$ has at least one root in $(1,2)$.

85. Let $f(x) = a_{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0 = a_{2n+1}x^{2n+1}\left(1 + \frac{a_{2n}}{a_{2n+1}x} + \cdots + \frac{a_0}{a_{2n+1}x^{2n+1}}\right)$, where $a_{2n+1} \ne 0$ and $x \ne 0$. Without loss of generality, let us assume that $a_{2n+1} > 0$. Observe that f is continuous, $f(x) < 0$ if x is negative and sufficiently large in absolute value, and $f(x) > 0$ if x is positive and sufficiently large in absolute value. Therefore, we can find numbers a and b with $a < b$ such that $f(a) < 0$ and $f(b) > 0$. Using the Intermediate Value Theorem, we conclude that there exists at least one number c in (a,b) such that $f(c) = 0$. Thus, the given equation has at least one real root.

90. f is continuous everywhere because it is a polynomial function. Since $f(0) = -1 < 0$ and $f(1) = 1^3 + 1 - 1 = 1 > 0$, the Intermediate Value Theorem implies that there is at least one number c in $(0,1)$ such that $f(c) = 0$. The number c is a zero of f.

96. False. Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$ Then $|f(x)| = 1$ is continuous everywhere, including at 0. But f is discontinuous at 0.

97. False. Let $f(x) = 1/x$. Then f is discontinuous at 0, and so is $f^2(x) = [f(x)]^2 = 1/x^2$.

98. False. Let $f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0 \\ 1 & \text{if } 0 \le x < 1 \end{cases}$ Then f is defined on $[-1,1]$, $f(-1) < 0$, and $f(1) > 0$, but f has no zeros in $(-1,1)$.

99. True. Write $g = (f+g) - f$. Since $f+g$ and f are both continuous, and the difference of two continuous functions is itself continuous, the result follows.

100. True. The interval $(2,4)$ is contained in the interval $(1,5)$. Since f is continuous on $(1,5)$, it is continuous at each point in $(1,5)$, including the entire interval $(2,4)$.