9.5

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n^2}{2n^2-1}$ is an alternating series, but $\lim_{n\to\infty} \frac{(-1)^{n-1}\, n^2}{2n^2-1}$ does not exist (the terms are close to $\pm\frac{1}{2}$ for large n, depending on the parity of n). Thus, the series diverges by the Divergence Test.

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{\sqrt{n^2+1}}$ is an alternating series with $a_n = \frac{n}{\sqrt{n^2+1}}$, but $\lim_{n\to\infty} \frac{(-1)^{n+1}\, n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{(-1)^{n+1}}{\sqrt{1+(1/n^2)}}$ does not exist (the terms are close to ± 1 for large n, depending on the parity of n). Thus, the series diverges by the Divergence Test.

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{ne^{-n}}$ is an alternating series with $a_n = \frac{1}{ne^{-n}}$, but because $\lim_{n\to\infty} \frac{1}{ne^{-n}} = \lim_{n\to\infty} \frac{e^n}{n} = \infty$, $a_n = \frac{1}{ne^{-n}}$ does not approach 0, and thus the given series diverges by the Divergence Test .

18. $\sum_{n=1}^{\infty} (-1)^n \cos\frac{\pi}{n}$. $\lim_{n\to\infty} (-1)^n \cos\frac{\pi}{n}$ does not exist because $\lim_{n\to\infty} \cos\frac{\pi}{n} = 1$, so the series diverges by the Divergence Test.

20. $\sum_{n=1}^{\infty} \frac{(-1)^n\, n!}{n^n}$ is an alternating series with $a_n = \frac{n!}{n^n}$. $\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n < 1 \Rightarrow$ $a_{n+1} < a_n$ and furthermore $\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{n\,(n-1)\,(n-2)\cdots(3)\,(2)\,(1)}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n\cdot n} \le \lim_{n\to\infty} \frac{1}{n} = 0$, so the AST implies that the given series converges.

9. 6

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$. $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$, so the series is absolutely convergent.

6. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is a convergent alternating series since $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$. But $\sum_{n=1}^{\infty} \left|\frac{(-1)^n n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by the Integral Test or the Limit Comparison Test using $\sum_{n=1}^{\infty} \frac{1}{n}$ as the test series. Therefore, the given series is conditionally convergent.

12. $\sum_{n=1}^{\infty} \frac{\cos(n+1)}{n\sqrt{n}}$. Observe that $\left|\frac{\cos(n+1)}{n\sqrt{n}}\right| \le \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$, we conclude by the Comparison Test that the given series is absolutely convergent.

15. $\sum_{n=1}^{\infty} \frac{2^n}{n!n}$. We use the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{2^{(n+1)}}{(n+1)!\,(n+1)} \cdot \frac{n!n}{2^n}\right] = \lim_{n\to\infty} \frac{2n}{(n+1)^2} = 0$, so the series converges.

19. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{2^n}$. Consider $\sum_{n=2}^{\infty} \left|\frac{(-1)^n \ln n}{2^n}\right| = \sum_{n=2}^{\infty} \frac{\ln n}{2^n}$. Since $\frac{\ln n}{2^n} < \frac{n}{2^n}$ and $\sum_{n=2}^{\infty} \frac{n}{2^n}$ converges (see Exercise 9.3.28), the Comparison Test implies that the given series converges absolutely.

9.7

4. Let $u_n = \frac{x^n}{n^2}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}}\right| = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^2\right]|x| = |x|$, so the radius of convergence is 1 and the series converges for $-1 < x < 1$. At $x = -1$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges, and at $x = 1$ it is $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which also converges. Thus, the interval of convergence is $[-1, 1]$.

13. Let $u_n = \frac{(-1)^n (x-3)^n}{\sqrt{n}}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-1)^n(x-3)^n}\right| = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}\,|x-3| = |x-3|$, so the radius of convergence is 1 and the series converges on $(2, 4)$. At $x = 2$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges, and at $x = 4$ it is $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges. Thus, the interval of convergence is $(2, 4]$.

19. Let $u_n = \frac{(-1)^n (x+2)^{2n+1}}{(2n+1)!}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+2)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n(x+2)^{2n+1}}\right| = \lim_{n\to\infty}\frac{|x+2|^2}{(2n+2)(2n+3)} = 0$ for any real x, so $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

21. Let $u_n = \frac{2^n (x+2)^n}{n^n}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x+2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n(x+2)^n}\right| = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)\left(\frac{n}{n+1}\right)^n|x+2| = 0$ for any real x since $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = \frac{1}{e}$. Thus, $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

41. Consider $\sum_{n=0}^{\infty}x^n$ for $|x| < 1$. Differentiating the geometric series gives $\frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=1}^{\infty}nx^{n-1}$, but $\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$, so $\sum_{n=1}^{\infty}nx^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ for $|x| < 1$.