Calculus Exam-2 (107.05.15)

- **1. (15%) Determine**

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2+1}{2^n}$$

is absolutely convergent, conditional convergent, or divergent. (p.778 , Example.4)

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○ Sol :

Use the ratio test with

$$a_n = (-1)^{n-1}\frac{n^2+1}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n((n+1)^2+1)}{2^{n+1}}\cdot\frac{2^n}{(-1)^{n-1}(n^2+1)}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n^2+2n+2}{n^2+1}\right) = \frac{1}{2} < 1$$

Hence the series is absolutely convergent.

- **2. (15%) Find the radius of convergence the interval of convergence of $\sum_{n=1}^{\infty}\frac{x^n}{n}$. (p.788 , Example.4)**

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○ Sol :

Let $u_n = x^n/n$ then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|x| = |x|$$

By the ratio test, the series converges if $|x| < 1$. And at the end points :

$$x = 1 \;\Rightarrow\; \sum_{n=1}^{\infty}\frac{1}{n} \;\Rightarrow\; \text{it is divergent}$$

$$x = -1 \;\Rightarrow\; \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \;\Rightarrow\; \text{it is convergent}$$

Hence the interval of convergence is $[-1, 1)$.

- **3. (15%) Let $f(x) = e^x$.**

Find the Maclaurin series of f, and determine its radius of convergence.

(p.795 , Example.1)

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∘ Sol :

In general $f^{(n)}(x) = e^x$ where $n \geq 1$. So

$$f(0) = 1 \ , \ f'(0) = 1 \ , \ f''(0) = 1 \ , \ \cdots \ f^{(n)}(0) = 1 \cdots$$

Therefore, the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

To determine the radius of convergence of the powers series

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0$$

Hence we conclude that the radius of convergence is $R = \infty$.

- **4. (15%) Find d^2y/dx^2 if $x = t^2 - 4$ and $y = t^3 - 3t$. (p.859 , Example.3)**

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∘ Sol :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{\frac{(2t)(6t)-2(3t^2-3)}{4t^2}}{2t} = \frac{6t^2+6}{8t^3} = \frac{3(t^2+1)}{4t^3}$$

- **5. (10%) Write $\vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$ as the sum of a vector parallel to $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ and a vector perpendicular to $\vec{a}$ (p.930 , Example.7)**

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○ Sol :

$$\begin{aligned} \text{proj}_{\vec{a}}\vec{b} &= \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\,\vec{a} = \frac{9}{6}(2\vec{i} - \vec{j} + \vec{k}) = 3\vec{i} - \frac{3}{2}\vec{j} + \frac{3}{2}\vec{k} \\ \vec{b} - \text{proj}_{\vec{a}}\vec{b} &= \frac{1}{2}\vec{j} + \frac{1}{2}\vec{k} \end{aligned}$$

Hence

$$\vec{b} = \text{proj}_{\vec{a}}\vec{b} + (\vec{b} - \text{proj}_{\vec{a}}\vec{b}) = \underbrace{\left(3\vec{i} - \frac{3}{2}\vec{j} + \frac{3}{2}\vec{k}\right)}_{\text{parallel to } \vec{a}} + \underbrace{\left(\frac{1}{2}\vec{j} + \frac{1}{2}\vec{k}\right)}_{\text{perpendicular to } \vec{a}}$$

- **6. (10%) Find area of triangular with vertices $P(3,-3,0)$, $Q(1,2,2)$ and $R(1,-2,5)$ (p.939 , Example.4)**

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○ Sol :

$$\overrightarrow{PQ} = -2\hat{i} + 5\hat{j} + 2\hat{k} \quad ; \quad \overrightarrow{PR} = -2\hat{i} + \hat{j} + 5\hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -2 & 5 & 2 \\ -2 & 1 & 5 \end{vmatrix} = 23\hat{i} + 6\hat{j} + 8\hat{k}$$

Hence area of triangular is

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{629}$$

- **7. (10%) Find an equation of the plane containing the points $P(3,-1,1)$, $Q(1,4,2)$ and $R(0,1,4)$ (p.950 , Example.6)**

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∘ Sol :

$$\overrightarrow{PQ} = -2\hat{\imath} + 5\hat{\jmath} + \hat{k} \quad ; \quad \overrightarrow{PR} = -3\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

The vector normal to this plane is

$$\hat{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -2 & 5 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 13\hat{\imath} + 3\hat{\jmath} + 11\hat{k}$$

Using the point $P(3,-1,1)$ on the plane, we have the equation of the plane

$$13(x-3) + 3(y+1) + 11(z-1) = 0 \ \Rightarrow \ 13x + 3y + 11z = 47$$

- **8. (10%)**

Find the antiderivative of $\vec{r}' = \cos t\ \hat{\imath} + e^{-t}\ \hat{\jmath} + \sqrt{t}\ \hat{k}$ satisfying the condition $\vec{r}(0) = \hat{\imath} + 2\hat{\jmath} + 3\hat{\jmath}$ (p.997 , Example.8)

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∘ Sol :

$$\vec{r}(t) = \int \vec{r}'(t)dt = \int \left(\cos t\ \hat{\imath} + e^{-t}\ \hat{\jmath} + \sqrt{t}\ \hat{k}\right)\ dt = \sin t\ \hat{\imath} - e^{-t}\ \hat{\jmath} + \frac{2}{3}t^{3/2}\ \hat{k} + \vec{C}$$

$$\vec{r}(0) = -\hat{\jmath} + \vec{C} = \hat{\imath} + 2\hat{\jmath} + 3\hat{\jmath} \ \Rightarrow \ \vec{C} = \hat{\imath} + 3\hat{\jmath} + 3\hat{\jmath}$$

Hence

$$\begin{aligned} \vec{r}(t) &= \sin t\ \hat{\imath} - e^{-t}\ \hat{\jmath} + \frac{2}{3}t^{3/2}\ \hat{k} + \left(\hat{\imath} + 3\hat{\jmath} + 3\hat{\jmath}\right) \\ &= (\sin t + 1)\hat{\imath} + (3 - e^{-t})\hat{\jmath} + \left(\frac{2}{3}t^{3/2} + 3\right)\hat{k} \end{aligned}$$
