13.9

7. $f(x,y) = xy \Rightarrow \nabla f(x,y) = y\mathbf{i} + x\mathbf{j}$ and $g(x,y) = x^2 + 4y^2 - 1 \Rightarrow \nabla g(x,y) = 2x\mathbf{i} + 8y\mathbf{j}$, so $\nabla f = \lambda \nabla g \Rightarrow$ $y\mathbf{i} + x\mathbf{j} = 2\lambda x\mathbf{i} + 8\lambda y\mathbf{j}$, and we solve $\left.\begin{aligned} y &= 2\lambda x \quad (1) \\ x &= 8\lambda y \quad (2) \\ x^2 + 4y^2 &= 1 \quad (3) \end{aligned}\right\}$ Substituting (1) into (2) gives $x = 8\lambda(2\lambda x) \Rightarrow$ $\lambda = \pm\frac{1}{4}$. [Note that $(x,y) \neq (0,0)$ since this violates (3).] If $\lambda = -\frac{1}{4}$, then (1) gives $y = -\frac{1}{2}x$ and so (3) gives $y = \pm\frac{\sqrt{2}}{4}$ and $x = \mp\frac{\sqrt{2}}{2}$; if $\lambda = \frac{1}{4}$, then (1) gives $y = \frac{1}{2}x$ and so (3) gives $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right)$.

(x,y)	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right)$
$f(x,y)$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

From the table, we see that f has a minimum value of $-\frac{1}{4}$ and a maximum value of $\frac{1}{4}$.

9. $f(x,y) = x^2 + xy + y^2 \Rightarrow \nabla f(x,y) = (2x+y)\mathbf{i} + (x+2y)\mathbf{j}$ and $g(x,y) = x^2 + y^2 - 8 \Rightarrow \nabla g(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$, so $\nabla f = \lambda \nabla g \Rightarrow (2x+y)\mathbf{i} + (x+2y)\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, and we solve $\left.\begin{aligned} 2x + y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ x^2 + y^2 &= 8 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= 2y(\lambda - 1) \\ y &= 2x(\lambda - 1) \end{aligned}\right\} \Rightarrow$ $y = \pm x$ (because $\lambda = 1 \Rightarrow x = y = 0$, violating the third equation). Substituting $y = \pm x$ into the third equation gives $x = \pm 2$ and $y = \pm 2$.

(x,y)	$(-2,-2)$	$(-2,2)$	$(2,-2)$	$(2,2)$
$f(x,y)$	12	4	4	12

From the table, we see that f has a minimum value of 4 and a maximum value of 12.

13. $f(x,y,z) = x + 2y - 2z \Rightarrow \nabla f(x,y,z) = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $g(x,y,z) = x^2 + 2y^2 + 4z^2 - 1 \Rightarrow$ $\nabla g(x,y,z) = 2x\mathbf{i} + 4y\mathbf{j} + 8z\mathbf{k}$, so $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j} + 8\lambda z\mathbf{k}$, and we solve $\left.\begin{aligned} 1 &= 2\lambda x \\ 2 &= 4\lambda y \\ -2 &= 8\lambda z \\ x^2 + 2y^2 + 4z^2 &= 1 \end{aligned}\right\}$ We see that $x = \dfrac{1}{2\lambda}$, $y = \dfrac{1}{2\lambda}$, and $z = -\dfrac{1}{4\lambda}$, which we substitute into the fourth equation: $\left(\dfrac{1}{2\lambda}\right)^2 + 2\left(\dfrac{1}{2\lambda}\right)^2 + 4\left(-\dfrac{1}{4\lambda}\right)^2 = 1 \Rightarrow \lambda = \pm 1$. If $\lambda = -1$, then $x = -\frac{1}{2}$, $y = -\frac{1}{2}$, and $z = \frac{1}{4}$; if $\lambda = 1$, then $x = \frac{1}{2}$, $y = \frac{1}{2}$, and $z = -\frac{1}{4}$. We see that the minimum value of f is $f\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}\right) = -2$ and the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}\right) = 2$.

15. $f(x,y,z) = xyz \Rightarrow \nabla f(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $g(x,y,z) = x^2 + 2y^2 + \frac{1}{2}z^2 = 6 \Rightarrow$ $\nabla g(x,y,z) = 2x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k}$, so $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j} + \lambda z\mathbf{k}$, and we solve $\left.\begin{aligned} yz &= 2\lambda x \\ xz &= 4\lambda y \\ xy &= \lambda z \\ x^2 + 2y^2 + \tfrac{1}{2}z^2 &= 6 \end{aligned}\right\}$ Solving the first three equations for λ gives $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{4y} = \dfrac{xy}{z}$, so $x^2 = 2y^2$ and $z^2 = 4y^2$. Substituting these into the last equation in the system gives $2y^2 + 2y^2 + \frac{1}{2}\left(4y^2\right) = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus, $x = \pm\sqrt{2}$ and $z = \pm 2$. Evaluating $f(x,y,z)$ at $\left(\pm\sqrt{2}, -1, -2\right)$, $\left(\pm\sqrt{2}, -1, 2\right)$, $\left(\pm\sqrt{2}, 1, -2\right)$, and $\left(\pm\sqrt{2}, 1, 2\right)$, we see that f has a minimum value of $-2\sqrt{2}$ and a maximum value of $2\sqrt{2}$.

17. $f(x,y,z) = 2x + y \Rightarrow \nabla f(x,y,z) = 2\mathbf{i} + \mathbf{j}$, $g(x,y,z) = x + y + z - 1 \Rightarrow \nabla g(x,y,z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $h(x,y,z) = y^2 + z^2 - 9 \Rightarrow \nabla h(x,y,z) = 2y\mathbf{j} + 2z\mathbf{k}$, so $\nabla f = \lambda \nabla g + \mu \nabla h$ along with the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$ give the system

$$\left.\begin{array}{rcll} 2 &=& \lambda & (1) \\ 1 &=& \lambda + 2\mu y & (2) \\ 0 &=& \lambda + 2\mu z & (3) \\ x + y + z &=& 1 & (4) \\ y^2 + z^2 &=& 9 & (5) \end{array}\right\}$$

From (1), (2), and (3), we obtain $\mu = -\frac{1}{2y} = -\frac{1}{z}$ $\Rightarrow z = 2y$. Substituting this into (5) gives $y^2 + (2y)^2 = 5y^2 = 9 \Leftrightarrow y = \pm\frac{3\sqrt{5}}{5} \Rightarrow z = \pm\frac{6\sqrt{5}}{5}$. Then (4) gives $x = 1 - \left(\pm\frac{3\sqrt{5}}{5}\right) - \left(\pm\frac{6\sqrt{5}}{5}\right) = \frac{5 \mp 9\sqrt{5}}{5}$, so f has minimum value $f\left(\frac{5 - 9\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}, \frac{6\sqrt{5}}{5}\right) = 2 - 3\sqrt{5}$ and maximum value $f\left(\frac{5 + 9\sqrt{5}}{5}, -\frac{3\sqrt{5}}{5}, -\frac{6\sqrt{5}}{5}\right) = 2 + 3\sqrt{5}$.

21. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(3x^2 + 2y^2 - 2x - 1\right) = 6x - 2 = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(3x^2 + 2y^2 - 2x - 1\right) = 4y \quad = 0 \end{array}\right\}$$ $\Rightarrow x = \frac{1}{3}$ and $y = 0$, so f has the critical point $\left(\frac{1}{3}, 0\right)$ in the disk $D = \left\{(x,y) \mid x^2 + y^2 \le 9\right\}$. Next, we use the method of Lagrange to find the critical points of f on the boundary of D. We write $g(x,y) = x^2 + y^2 - 9 = 0$. Then $\nabla f(x,y) = (6x - 2)\mathbf{i} + 4y\mathbf{j}$ and $\nabla g(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ and the constraint equation $g(x,y) = 0$ give the system

$$\left.\begin{array}{rcll} 6x - 2 &=& 2\lambda x & (1) \\ 4y &=& 2\lambda y & (2) \\ x^2 + y^2 &=& 9 & (3) \end{array}\right\}$$

Equation (2) gives $y = 0$ or $\lambda = 2$. If $y = 0$, then (3) gives $x = \pm 3$; if $\lambda = 2$, then (1) gives $x = 1$. Substituting this value of x into (3) gives $y = \pm 2\sqrt{2}$.

(x,y)	$\left(\frac{1}{3}, 0\right)$	$(-3, 0)$	$(3, 0)$	$\left(1, -2\sqrt{2}\right)$	$\left(1, 2\sqrt{2}\right)$
$f(x,y)$	$-\frac{4}{3}$	32	20	16	16

From the table, we see that f has a minimum value of $f\left(\frac{1}{3}, 0\right) = -\frac{4}{3}$ and a maximum value of $f(-3, 0) = 32$.

23. The distance from the origin to a point (x,y,z) on the plane $x + 2y + z = 4$ is $D = \sqrt{x^2 + y^2 + z^2}$. To minimize D, we just need to minimize $f(x,y,z) = D^2 = x^2 + y^2 + z^2$ subject to $g(x,y,z) = x + 2y + z - 4 = 0$. $\nabla f(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g(x,y,z) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, so $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 0$ give the system

$$\left.\begin{array}{rcl} 2x &=& \lambda \\ 2y &=& 2\lambda \\ 2z &=& \lambda \\ x + 2y + z &=& 4 \end{array}\right\} \Leftrightarrow \left.\begin{array}{rcl} x &=& \frac{1}{2}\lambda \\ y &=& \lambda \\ z &=& \frac{1}{2}\lambda \\ x + 2y + z &=& 4 \end{array}\right\}$$

$\Rightarrow \frac{1}{2}\lambda + 2\lambda + \frac{1}{2}\lambda = 4$, giving $\lambda = \frac{4}{3}$. Therefore, the required point is $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$.

13.10

3. $\int_0^2 \int_1^4 y\sqrt{x}\,dy\,dx = \int_0^2 \left[\int_1^4 yx^{1/2}\,dy\right] dx = \int_0^2 \left[\frac{1}{2}y^2x^{1/2}\right]_{y=1}^{y=4} dx = \int_0^2 \frac{15}{2}x^{1/2}\,dx = 5x^{3/2}\Big|_0^2 = 10\sqrt{2}$

6. $\int_0^{\pi/2} \int_0^{\ln 2} e^{-x}\sin y\,dx\,dy = \int_0^{\pi/2} \left[\int_0^{\ln 2} e^{-x}\sin y\,dx\right] dy = \int_0^{\pi/2} \left[-e^{-x}\sin y\right]_{x=0}^{x=\ln 2} dy = \int_0^{\pi/2} \frac{1}{2}\sin y\,dy$

$= -\frac{1}{2}\cos y\Big|_0^{\pi/2} = \frac{1}{2}$

11. $\int_{-1}^1 \int_x^{2x} e^{x+y}\,dy\,dx = \int_{-1}^1 \left[\int_x^{2x} e^{x+y}\,dy\right] dx = \int_{-1}^1 \left[e^{x+y}\right]_{y=x}^{y=2x} dx = \int_{-1}^1 \left(e^{3x} - e^{2x}\right) dx = \left(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\right)\Big|_{-1}^1$

$= \left(\frac{1}{3}e^3 - \frac{1}{2}e^2\right) - \left(\frac{1}{3}e^{-3} - \frac{1}{2}e^{-2}\right) = \dfrac{2e^6 - 3e^5 + 3e - 2}{6e^3}$

12. $\displaystyle\int_0^{\pi} \int_{e^{-2x}}^{e^{\cos x}} \frac{\ln y}{y}\,dy\,dx = \int_0^{\pi} \left[\int_{e^{-2x}}^{e^{\cos x}} \frac{\ln y}{y}\,dy\right] dx = \int_0^{\pi} \left[\frac{1}{2}(\ln y)^2\right]_{y=e^{-2x}}^{y=e^{\cos x}} dx = \frac{1}{2}\int_0^{\pi} \left[\left(\ln e^{\cos x}\right)^2 - \left(\ln e^{-2x}\right)^2\right] dx$

$= \frac{1}{2}\int_0^{\pi} \left(\cos^2 x - 4x^2\right) dx = \frac{1}{2}\int_0^{\pi} \left[\frac{1}{2} + \frac{1}{2}\cos 2x - 4x^2\right] dx$

$= \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x - \frac{4}{3}x^3\right)\Big|_0^{\pi} = \frac{1}{12}\pi\left(3 - 8\pi^2\right)$

15. $\iint_R (x\cos y + y\sin x)\,dA = \int_0^{\pi/4} \int_0^{\pi/2} (x\cos y + y\sin x)\,dx\,dy = \int_0^{\pi/4} \left[\frac{1}{2}x^2\cos y - y\cos x\right]_{x=0}^{x=\pi/2} dy$

$= \int_0^{\pi/4} \left(\frac{1}{8}\pi^2\cos y + y\right) dy = \left(\frac{1}{8}\pi^2\sin y + \frac{1}{2}y^2\right)\Big|_0^{\pi/4} = \frac{1}{32}\pi^2\left(2\sqrt{2} + 1\right)$

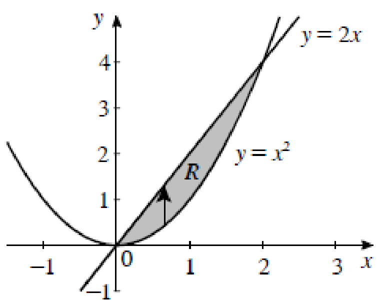
16. $\iint_R ye^{xy}\,dA = \int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 \left(e^y - 1\right) dy = \left(e^y - y\right)\Big|_0^1 = e - 2$

19. $\iint_R \left(x^3 + 2y\right) dA = \int_0^2 \int_{x^2}^{2x} \left(x^3 + 2y\right) dy\,dx$

$= \int_0^2 \left[x^3y + y^2\right]_{y=x^2}^{y=2x} dx$

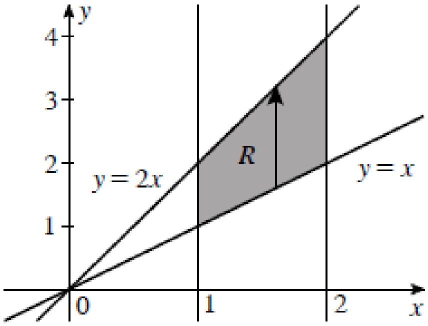
$= \int_0^2 \left(x^4 + 4x^2 - x^5\right) dx$

$= \left(\frac{1}{5}x^5 + \frac{4}{3}x^3 - \frac{1}{6}x^6\right)\Big|_0^2 = \frac{32}{5}$



25. $\iint_R x^2y\,dA = \int_1^2 \int_x^{2x} x^2y\,dy\,dx = \int_1^2 \left[\frac{1}{2}x^2y^2\right]_{y=x}^{y=2x} dx$

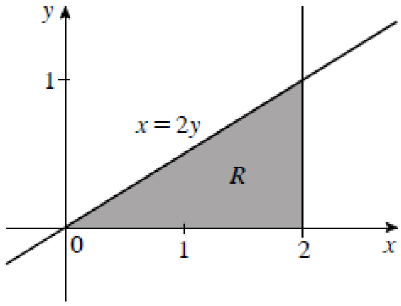
$= \frac{1}{2}\int_1^2 3x^4\,dx = \frac{3}{10}x^5\Big|_1^2 = \frac{93}{10}$



55. $\int_0^1 \int_{2y}^2 e^{-x^2}\,dx\,dy = \int_0^2 \int_0^{x/2} e^{-x^2}\,dy\,dx = \int_0^2 \left[ye^{-x^2}\right]_{y=0}^{y=x/2} dx$

$= \frac{1}{2}\int_0^2 xe^{-x^2}\,dx = -\frac{1}{4}e^{-x^2}\Big|_0^2 = \frac{1}{4}\left(1 - e^{-4}\right)$

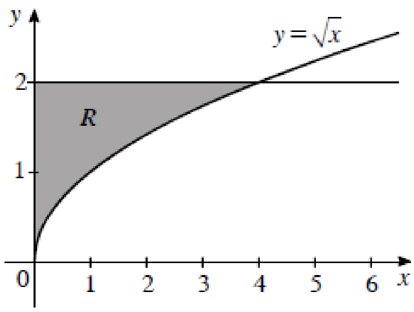
$= \dfrac{e^4 - 1}{4e^4}$



57. $\int_0^4 \int_{\sqrt{x}}^2 \sin y^3\,dy\,dx = \int_0^2 \int_0^{y^2} \sin y^3\,dx\,dy = \int_0^2 \left[x\sin y^3\right]_{x=0}^{x=y^2} dy$

$= \int_0^2 y^2\sin y^3\,dy = -\frac{1}{3}\cos y^3\Big|_0^2$

$= \dfrac{1 - \cos 8}{3}$



59. $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{\sqrt{x^3+1}}\,dx\,dy = \int_0^2 \int_0^{x^2} \frac{1}{\sqrt{x^3+1}}\,dy\,dx = \int_0^2 \left[\frac{y}{\sqrt{x^3+1}}\right]_{y=0}^{y=x^2} dx$

$\displaystyle= \int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{2}{3}\left(x^3+1\right)^{1/2}\Big|_0^2$

$= \frac{2}{3}(3-1) = \frac{4}{3}$