13.7

6. $F(x,y) = x^4 - x^2 + y^2 \Rightarrow \nabla F\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) = \left[\left(4x^3 - 2x\right)\mathbf{i} + 2y\mathbf{j}\right]_{\left(1/2, \sqrt{3}/4\right)} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ is normal to the level curve $F(x,y) = x^4 - x^2 + y^2 = 0$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$. So the slope of the required normal line is $m = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$ and an equation of the normal line is $y - \frac{\sqrt{3}}{4} = -\sqrt{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = -\sqrt{3}x + \frac{3\sqrt{3}}{4}$. The slope of the required tangent line is $m = -\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$, and so an equation of the tangent line is $y - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{12}$.

8. $F(x,y) = 2x + y - e^{x-y} \Rightarrow \nabla F(1,1) = \left[\left(2 - e^{x-y}\right)\mathbf{i} + \left(1 + e^{x-y}\right)\mathbf{j}\right]_{(1,1)} = \mathbf{i} + 2\mathbf{j}$ is normal to the level curve $F(x,y) = 2x + y - e^{x-y} = 2$ at $(1,1)$. From this, we see that the slope of the required normal line is $m = \frac{2}{1} = 2$ and so an equation of the normal line is $y - 1 = 2(x-1) \Leftrightarrow y = 2x - 1$. The slope of the required tangent line is $m = -\frac{1}{2}$ and so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

15. $F(x,y,z) = x^2 + 4y^2 + 9z^2 - 17 \Rightarrow \nabla F(2,1,1) = (2x\mathbf{i} + 8y\mathbf{j} + 18z\mathbf{k})|_{(2,1,1)} = 4\mathbf{i} + 8\mathbf{j} + 18\mathbf{k}$, so an equation of the tangent plane at $(2,1,1)$ is $4(x-2) + 8(y-1) + 18(z-1) = 0 \Leftrightarrow 2x + 4y + 9z = 17$. Equations of the normal line passing through $(2,1,1)$ are $\dfrac{x-2}{4} = \dfrac{y-1}{8} = \dfrac{z-1}{18} \Leftrightarrow \dfrac{x-2}{2} = \dfrac{y-1}{4} = \dfrac{z-1}{9}$.

19. $F(x,y,z) = xy + yz + xz - 11 = 0 \Rightarrow \nabla F(1,2,3) = \left[(y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}\right]_{(1,2,3)} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, so an equation of the tangent plane at $(1,2,3)$ is $5(x-1) + 4(y-2) + 3(z-3) = 0 \Leftrightarrow 5x + 4y + 3z = 22$. Equations of the normal line passing through $(1,2,3)$ are $\dfrac{x-1}{5} = \dfrac{y-2}{4} = \dfrac{z-3}{3}$.

26. $F(x,y,z) = e^x \sin \pi y - z = 0 \Rightarrow \nabla F(0,1,0) = \left(e^x \sin \pi y\,\mathbf{i} + \pi e^x \cos \pi y\,\mathbf{j} - \mathbf{k}\right)|_{(0,1,0)} = -\pi\mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $(0,1,0)$ is $-\pi(y-1) - z = 0 \Leftrightarrow \pi y + z = \pi$. Equations of the normal line passing through $(0,1,0)$ are $x = 0$, $\dfrac{y-1}{-\pi} = \dfrac{z}{-1} \Leftrightarrow x = 0$, $\dfrac{y-1}{\pi} = z$.

28. $F(x,y,z) = \ln \dfrac{x}{y} - z = \ln x - \ln y - z = 0 \Rightarrow \nabla F(2,2,0) = \left(\frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j} - \mathbf{k}\right)\Big|_{(2,2,0)} = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $(2,2,0)$ is $\frac{1}{2}(x-2) - \frac{1}{2}(y-2) - z = 0 \Leftrightarrow x - y - 2z = 0$. Equations of the normal line passing through $(2,2,0)$ are $\dfrac{x-2}{1/2} = \dfrac{y-2}{-1/2} = \dfrac{z}{-1} \Leftrightarrow x - 2 = \dfrac{y-2}{-1} = \dfrac{z}{-2}$.

31. $F(x,y,z) = \sin xy + 3z - 3 = 0 \Rightarrow \nabla F(0,3,1) = (y \cos xy\,\mathbf{i} + x \cos xy\,\mathbf{j} + 3\mathbf{k})|_{(0,3,1)} = 3\mathbf{i} + 3\mathbf{k} = 3(\mathbf{i} + \mathbf{k})$, so an equation of the tangent plane at $(0,3,1)$ is $(x-0) + (z-1) = 0 \Leftrightarrow x + z = 1$. Equations of the normal line passing through $(0,3,1)$ are $\dfrac{x}{1} = \dfrac{z-1}{1}$, $y = 3$.

37. We write $F(x,y,z) = x^2 + y^2 + z^2 - 14$. Then the normal to the tangent plane to the sphere at the point (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = 2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}$. Since the tangent plane is parallel to the plane $x + 2y + 3z = 12$ whose normal is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, we see that $x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} = c(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, where c is a constant. This equation implies that $x_0 = c$, $y_0 = 2c$, and $z_0 = 3c$, and substituting these values into the equation of the sphere gives $c^2 + (2c)^2 + (3c)^2 = 14 \Leftrightarrow 14c^2 = 14 \Leftrightarrow c = \pm 1$. Thus, the required points are $(-1,-2,-3)$ and $(1,2,3)$.

13.8

3. To find the critical points of f, we solve the system

$$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(-x^2-3y^2+4x-6y+8\right) = -2x+4=0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(-x^2-3y^2+4x-6y+8\right) = -6y-6=0 \end{array}\right\}$$ obtaining the sole critical point $(2,-1)$ of f.

Next, we use the SDT on the critical point: $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = (-2)(-6) - 0^2 = 12$. Since $D(2,-1) = 12 > 0$ and $f_{xx}(2,-1) = -2 < 0$, the point $(2,-1)$ gives a relative maximum of f with value $f(2,-1) = -2^2 - 3(-1)^2 + 4(2) - 6(-1) + 8 = 15$.

7. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(2x^2+y^2-2xy-8x-2y+2\right) = 4x-2y-8=0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(2x^2+y^2-2xy-8x-2y+2\right) = 2y-2x-2=0 \end{array}\right\} \Rightarrow x = 5,\ y = 6,$$ so $(5,6)$ is the sole critical point of f. Next, $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = 4\cdot 2 - (-2)^2 = 4$. Since $D(5,6) = 4 > 0$ and $f_{xx}(5,6) = 4 > 0$, the point $(5,6)$ gives a relative minimum of f with value $f(5,6) = 2(5)^2 + 6^2 - 2(5)(6) - 8(5) - 2(6) + 2 = -24$.

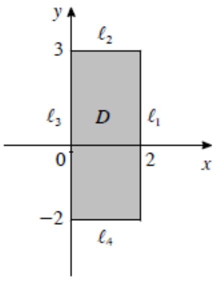
9. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2+2y^2+x^2y+3\right) = 2x+2xy=0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2+2y^2+x^2y+3\right) = 4y+x^2=0 \end{array}\right\}$$ The first equation gives $x = 0$ or $y = -1$. Substituting $x = 0$ into the second equation gives $y = 0$; substituting $y = -1$ into the second equation gives $x = \pm 2$. Thus, f has critical points $(0,0)$, $(-2,-1)$, and $(2,-1)$. Next,
$D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = (2+2y)(4) - (2x)^2 = -4x^2 + 8y + 8$.
At $(0,0)$: $D(0,0) = 8 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $(0,0)$ gives a relative minimum of f with value $f(0,0) = 3$.
At $(-2,-1)$: $D(-2,-1) = -4(-2)^2 + 8(-1) + 8 = -16 < 0$, so $(-2,-1,5)$ is a saddle point of f.
At $(2,-1)$: $D(2,-1) = -4(2)^2 + 8(-1) + 8 = -16 < 0$, so $(2,-1,5)$ is also a saddle point of f.

11. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2+5y^2+x^2y+2y^3\right) = 2x+2xy = 2x(y+1)=0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2+5y^2+x^2y+2y^3\right) = 10y+x^2+6y^2=0 \end{array}\right\}$$ The first equation gives $x = 0$ or $y = -1$.
Substituting $x = 0$ into the second equation gives $10y + 6y^2 = 2y(3y+5) = 0 \Rightarrow y = 0$ or $-\frac{5}{3}$; substituting $y = -1$ into the second equation gives $x^2 - 4 = 0 \Rightarrow x = \pm 2$. Thus, f has critical points $(0,0)$, $\left(0,-\frac{5}{3}\right)$, $(-2,-1)$, and $(2,-1)$. Next,
$D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = 2(y+1)(10+12y) - (2x)^2 = 4(y+1)(5+6y) - 4x^2$.
At $(0,0)$: $D(0,0) = 4(1)(5) - 0 = 20 > 0$ and $f_{xx}(0,0) = 2 > 0$, so f has a relative minimum at $(0,0)$ with value $f(0,0) = 0$.
At $\left(0,-\frac{5}{3}\right)$: $D\left(0,-\frac{5}{3}\right) = 4\left(-\frac{5}{3}+1\right)\left[5+6\left(-\frac{5}{3}\right)\right] - 4(0) = \frac{40}{3} > 0$ and $f_{xx}\left(0,-\frac{5}{3}\right) = 2\left(-\frac{5}{3}+1\right) = -\frac{4}{3} < 0$, so f has a relative maximum at $\left(0,-\frac{5}{3}\right)$ with value $f\left(0,-\frac{5}{3}\right) = \frac{125}{27}$.
At $(-2,-1)$: $D(-2,-1) = 0 - 4(-2)^2 = -16 < 0$, and so $(-2,-1,3)$ is a saddle point of f.
At $(2,-1)$: $D(2,-1) = 0 - 4(2)^2 = -16 < 0$, and so $(2,-1,3)$ is also a saddle point of f.

17. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(e^{-x^2-y^2}\right) = -2xe^{-x^2-y^2} \\ f_y(x,y) = \frac{\partial}{\partial y}\left(e^{-x^2-y^2}\right) = -2ye^{-x^2-y^2} \end{array}\right\} \Rightarrow f_{xx}(x,y) = 2\left(2x^2-1\right)e^{-x^2-y^2},\ f_{xy}(x,y) = 4xye^{-x^2-y^2},$$ and $f_{yy}(x,y) = 2\left(2y^2-1\right)e^{-x^2-y^2}$. Setting $\left.\begin{array}{l} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{array}\right\} \Rightarrow x = 0$ and $y = 0$, so $(0,0)$ is the sole critical point of f.
Next, $D(0,0) = f_{xx}(0,0)\,f_{yy}(0,0) - f_{xy}^2(0,0) = (-2)(-2) - 0^2 = 4 > 0$ and $f_{xx}(0,0) = -2 < 0$, we see that f has a relative maximum at $(0,0)$ with value $f(0,0) = 1$.

33. Since $f_x(x,y) = \frac{\partial}{\partial x}(2x + 3y - 6) = 2$ and

$f_y(x,y) = \frac{\partial}{\partial y}(2x + 3y - 6) = 3$ are never equal to 0, f has no critical point on D.



On ℓ_1, $x = 2$ and $y = y$, so $g(y) = f(2,y) = 3y - 2$ for $-2 \le y \le 3$.

We see that g has an absolute minimum value of -8 at $(2,-2)$ and an absolute maximum value of 7 at $(2,3)$.

On ℓ_2, $x = x$ and $y = 3$, so $h(x) = f(x,3) = 2x + 3$ for $0 \le x \le 2$. We see that h has an absolute minimum value of 3 at $(0,3)$ and an absolute maximum value of 7 at $(2,3)$.

On ℓ_3, $x = 0$ and $y = y$, so $s(y) = f(0,y) = 3y - 6$ for $-2 \le y \le 3$. We see that s has an absolute minimum value of -12 at $(0,-2)$ and an absolute maximum value of 3 at $(0,3)$.

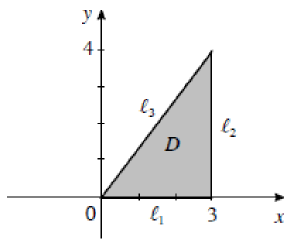
On ℓ_4, $x = x$ and $y = -2$, so $t(x) = f(x,-2) = 2x - 12$ for $0 \le x \le 2$. We see that t has an absolute minimum value of -12 at $(0,-2)$ and an absolute maximum value of -8 at $(2,-2)$.

The extreme values of f on each boundary of D are summarized below.

	ℓ_1		ℓ_2		ℓ_3		ℓ_4	
(x,y)	$(2,-2)$	$(2,3)$	$(0,3)$	$(2,3)$	$(0,-2)$	$(0,3)$	$(0,-2)$	$(2,-2)$
Extreme value	-8	7	3	7	-12	3	-12	-8

We see that f has an absolute minimum value of $f(0,-2) = -12$ and an absolute maximum value of $f(2,3) = 7$ on D.

35. $f_x(x,y) = 3$ and $f_y(x,y) = 4$, so f has no critical point.



On ℓ_1, $x = x$ and $y = 0$, so $g(y) = f(x,0) = 3x - 12$ for $0 \le x \le 3$.

We see that f has an absolute minimum value of -12 and an absolute maximum value of -3 on ℓ_1.

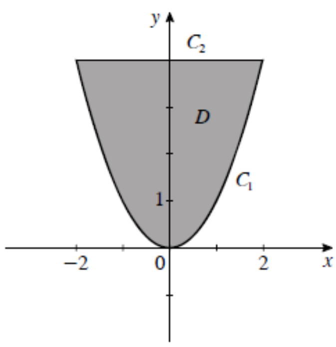
On ℓ_2, $x = 3$ and $y = y$, so $h(y) = f(3,y) = 4y - 3$ for $0 \le y \le 4$. We see that f has an absolute minimum value of -3 and an absolute maximum value of 13 on ℓ_2.

On ℓ_3, $y = \frac{4}{3}x$, so $s(x) = f\left(x, \frac{4}{3}x\right) = 3x + 4\left(\frac{4}{3}x\right) - 12 = \frac{25}{3}x - 12$ for $0 \le x \le 3$. We see that f has an absolute minimum value of -12 and an absolute maximum value of 13 on ℓ_3.

From these calculations, we see that f has an absolute minimum value of -12 and an absolute maximum value of 13 on D.

37. $$\left.\begin{aligned} f_x(x,y) &= \tfrac{\partial}{\partial x}\left(xy - x^2\right) = y - 2x = 0 \\ f_y(x,y) &= \tfrac{\partial}{\partial y}\left(xy - x^2\right) = x = 0 \end{aligned}\right\} \Rightarrow x = 0,\ y = 0, \text{ so } f \text{ has}$$

no critical point in the interior of D.



On C_1, $y = x^2$, so $g(x) = f\left(x, x^2\right) = x^3 - x^2$ for $-2 \le x \le 2$.

$g'(x) = 3x^2 - 2x = x(3x - 2) = 0 \Rightarrow x = 0$ or $x = \frac{2}{3}$, so 0 and $\frac{2}{3}$ are critical numbers of g on $(-2,2)$.

x	-2	0	$\frac{2}{3}$	2
$g(x)$	-12	0	$-\frac{4}{27}$	4

From the table, we see that f has an absolute minimum value of -12 and an absolute maximum value of 4 on C_1.

On C_2, $x = x$ and $y = 4$, so $h(x) = f(x,4) = 4x - x^2$ for $-2 \le x \le 2$.

$h'(x) = 4 - 2x = 0 \Rightarrow x = 2$, an endpoint. We find $h(-2) = -12$ and $g(2) = 4$, so f has an absolute minimum value of -12 and an absolute maximum value of 4 on C_2.

We conclude that f has an absolute minimum value of -12 and an absolute maximum value of 4 on D.