Calculus Exam-3 (107.01.16)

- **1. (15%) (Section(4.5)-Example.4)**

  If $\boldsymbol{y = \int_0^{x^3} \cos t^2 dt}$ what is $\boldsymbol{dy/dx =}$??.

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○ Sol :

$$\text{Let } u = x^3 \;\Rightarrow\; \frac{du}{dx} = 3x^2$$

Using the chain rule and the Fundamental Theorem of Calculus, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du} \int_0^u \cos t^2 dt \right) \cdot (3x^2) = \cos u^2 \cdot (3x^2) = 3x^2 \cos(x^6)$$

- **2. (15%) (Exercises(6.2)-(56))**

Suppose that g is the inverse of a differentiable function f and $H = g \circ g$.

If $f(4) = 3$, $g(4) = 5$, $f'(4) = 1/2$, and $f'(5) = 2$ find $H'(3)$.

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○ Sol :

$$\frac{d}{dx} H(x) = \frac{d}{dx} [g(g(x))] = g'(g(x)) \cdot g'(x) \;\Rightarrow\; H'(3) = g'(g(3)) \cdot g'(3)$$

And

$$g = f^{-1} \;\Rightarrow\; g(3) = f^{-1}(3) = 4 \quad ; \quad \text{and } \; g'(x) = \frac{1}{f'(g(x))}$$

then

$$H'(3) = g'(g(3)) \cdot g'(3) = g'(4) \cdot g'(3) = \frac{1}{f'(g(4))} \cdot \frac{1}{f'(g(3))} = \frac{1}{f'(5)} \cdot \frac{1}{f'(4)} = \frac{1}{2} \cdot 2 = 1$$

- **3. (15%) (Section(6.4)-Example.4)**

  Find the derivative of $\boldsymbol{f(x) = x^x}$

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○ Sol :

$$x^x = e^{x\ln x} \;\Rightarrow\; \frac{d}{dx}f(x) = e^{x\ln x}\cdot(\ln x + x\cdot\frac{1}{x}) = x^x\cdot(1+\ln x)$$

- **4. (15%) (see Section(6.4) – The definition of the number e as a limit)**

  Evaluate $\displaystyle\lim_{\boldsymbol{x\to\infty}}\left(\boldsymbol{1+\frac{1}{x}}\right)^{\boldsymbol{x}}$

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○ Sol :

$$\text{Let}\quad \left(1+\frac{1}{x}\right)^x = A \;\Rightarrow\; \ln A = x\ln\left(1+\frac{1}{x}\right)$$

$$\Rightarrow\; \lim_{x\to\infty}\ln A = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\;,\ \text{let } 1/x = k \;\Rightarrow\; \lim_{x\to\infty}\ln A = \lim_{k\to 0}\frac{\ln(1+k)}{k}$$

By Hopital's Rule we have

$$\lim_{x\to\infty}\ln A = \lim_{k\to 0}\frac{\ln(1+k)}{k} = \lim_{k\to 0}\frac{1}{1+k} = 1$$

And

$$\because \lim_{x\to\infty}\ln A = \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = 1 \;\Rightarrow\; \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] = 1 \;\Rightarrow\; \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

- **5. (10%) (Section(5.1)-Example.3 and Example.4)**

Find the area of the region bounded by the graph of $x = y^2$ and $y = x - 2$.

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- Sol :

The bounded region is shown in Fig.(11) of (5.1) in text book.

We could find the area by integrating with respect to y.

Find the intersect points

$$y^2 = y + 2 \;\Rightarrow\; y^2 - y - 2 = 0 \;\Rightarrow\; (y-2)(y+1) = 0 \;\Rightarrow\; y = 2\;,\; -1$$

Hence the area is

$$\begin{aligned} A &= \int_{-1}^{2} (y+2) - y^2 \; dy \\ &= \int_{-1}^{2} (-y^2 + y + 2)\; dy \\ &= \left(-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y \right)\Bigg|_{-1}^{2} = \left(-\frac{8}{3} + 2 + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) = \frac{9}{2} \end{aligned}$$

- **6. (10%) (Section(5.2)-Example.4)**

Find the volume of the solid obtained by revolving the region bounded $y = \sqrt{x}$ and $y = x$ about the x-axis.

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- Sol :

The bounded region is shown in Fig.(12) of (5.2) in text book.

The intersect points : ($x > 0$)

$$x = \sqrt{x} \;\Rightarrow\; x^2 - x = 0 \;\Rightarrow\; x = 0, 1$$

And

$$\Delta V = \pi((\sqrt{x})^2 - x^2)\Delta x = \pi(x - x^2)\Delta x$$

Hence

$$Volume = \int \Delta V = \pi \int_0^1 (x - x^2)\; dx = \pi \left(\frac{1}{2}x^2 - \frac{1}{3}x^2 \right)\Bigg|_0^1 = \pi(\frac{1}{2} - \frac{1}{3}) = \frac{\pi}{6}$$

- **7. (10%) (Section(5.3)-Example.3)**

Let R be the region bounded by the graph of $x = -y^2 + 6y$ and $x = 0$. Find the volume of the solid obtained by revolving R about the x-axis.

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○ Sol :

The bounded region is shown in Fig.(9) of (5.3) in text book.

Find the intersect points with the y-axis

$$-y^2 + 6y = 0 \;\Rightarrow\; y(-y+6) = 0 \;\Rightarrow\; y = 0\ ,\ 6$$

And

$$\Delta V = 2\pi y(-y^2 + 6y)\Delta y = 2\pi(-y^3 + 6y^2)\Delta y$$

Hence

$$V = \int \Delta V = \int_0^6 \; 2\pi(-y^3 + 6y^2)\; dy = 2\pi \left(-\frac{1}{4}y^4 + 2y^3 \right) \Bigg|_0^6 = 2\pi(-324 + 432) = 216\pi$$

- **8. (10%) (Section(6.5)-Example.6)**

$$\textbf{Find} \;\; \int \frac{\mathbf{1}}{\sqrt{\mathbf{1} - \mathbf{9}\boldsymbol{x}^{\mathbf{2}}}} \; \boldsymbol{dx}$$

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○ Sol :

$$\text{Let } u = 3x \;\Rightarrow\; du = 3dx \;\Rightarrow\; dx = \frac{1}{3}du$$

then

$$\int \frac{1}{\sqrt{1 - 9x^2}}\; dx = \frac{1}{3} \int \frac{1}{\sqrt{1 - u^2}}\; du = \frac{1}{3}\sin^{-1}(u) + C = \frac{1}{3}\sin^{-1}(3x) + C$$
