6.1 The Natural Logarithmic Function

28. $g(x) = \ln\left(x^2+4\right)^2 = 2\ln\left(x^2+4\right) \Rightarrow g'(x) = 2\frac{d}{dx}\ln\left(x^2+4\right) = \frac{2(2x)}{x^2+4} = \frac{4x}{x^2+4}$

30. $y = \sqrt{\ln x} = (\ln x)^{1/2} \Rightarrow y' = \frac{1}{2}(\ln x)^{-1/2}\cdot\frac{d}{dx}\ln x = \frac{1}{2x\sqrt{\ln x}}$

40. $f(x) = \ln(x\ln(x+2)) = \ln x + \ln(\ln(x+2))$, so $f'(x) = \frac{1}{x} + \frac{\frac{1}{x+2}}{\ln(x+2)} = \frac{(x+2)\ln(x+2)+x}{x(x+2)\ln(x+2)}$.

42. $h'(t) = \frac{d}{dt}[t\sin(\ln 2t)] = \sin(\ln 2t) + t\frac{d}{dt}[\sin(\ln 2t)] = \sin(\ln 2t) + t\cos(\ln 2t)\cdot\frac{1}{t} = \sin(\ln 2t)+\cos(\ln 2t)$

51. $\ln\frac{x}{y} + x - y^2 = 0 \Rightarrow \ln x - \ln y + x - y^2 = 0$. Differentiating, we obtain $\frac{1}{x} - \frac{y'}{y} + 1 - 2yy' = 0 \Rightarrow y'\left(\frac{1}{y}+2y\right) = 1 + \frac{1}{x}$
$\Rightarrow y' = \frac{(x+1)y}{x\left(2y^2+1\right)}$.

52. $\ln(x+y) - \cos y - x^2 = 0 \Rightarrow \frac{1+y'}{x+y} + (\sin y)y' - 2x = 0 \Rightarrow \frac{1}{x+y} - 2x + \left(\frac{1}{x+y}+\sin y\right)y' = 0 \Rightarrow$
$\frac{1-2x(x+y)}{x+y} + \frac{1+(x+y)\sin y}{x+y}y' = 0 \Rightarrow y' = \frac{2x(x+y)-1}{1+(x+y)\sin y}$

56. $y - \ln\left(x^2+y^2\right) = 0 \Rightarrow y' - \frac{2x+2yy'}{x^2+y^2} = 0$. Substituting $x=1$ and $y=0$ into the equation gives $y' - \frac{2+0}{1+0} = 0$ or $y' = 2$, the slope of the required tangent line. An equation is $y - 0 = 2(x-1)$ or $y = 2x-2$.

66. $y = \frac{x^2\sqrt{2x-4}}{(x+1)^2} \Rightarrow \ln y = 2\ln x + \frac{1}{2}\ln(2x-4) - 2\ln(x+1) \Rightarrow \frac{y'}{y} = \frac{2}{x} + \frac{1}{2x-4} - \frac{2}{x+1} = \frac{x^2+5x-8}{2x(x-2)(x+1)} \Rightarrow$
$y' = \frac{x\left(x^2+5x-8\right)}{\sqrt{2x-4}\,(x+1)^3}$

69. Taking logarithms of both sides gives $\ln y = \ln x^x = x\ln x$, so $\frac{y'}{y} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x \Rightarrow y' = (\ln x+1)x^x$.
Therefore, $y'' = (\ln x+1)\frac{d}{dx}x^x + x^x\frac{d}{dx}(\ln x+1) = (\ln x+1)(\ln x+1)x^x + x^x\left(\frac{1}{x}\right) = \left[x(\ln x+1)^2+1\right]x^{x-1}$.

72. Let $u = 2x+3$. Then $du = 2\,dx$, so $\int\frac{1}{2x+3}\,dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x+3| + C$.

74. $\int_1^3 \frac{x^2-x+3}{x}\,dx = \int_1^3 \left(x-1+\frac{3}{x}\right)dx = \left(\frac{1}{2}x^2 - x + 3\ln|x|\right)\Big|_1^3 = 2 + 3\ln 3$

76. Let $u = \ln x$. Then $du = \frac{dx}{x}$, $x = 1 \Rightarrow u = 0$, and $x = 3 \Rightarrow u = \ln 3$. Thus,
$\int_1^3 \frac{\ln x}{x}\,dx = \int_0^{\ln 3} u\,du = \frac{1}{2}u^2\Big|_0^{\ln 3} = \frac{1}{2}(\ln 3)^2$.

78. Let $u = 1 + \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{\sqrt{1+\ln x}}{x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+\ln x)^{3/2} + C$.

80. Let $u = 4 - \tan 3x$. Then $du = -3\sec^2 3x\,dx$, so $\int \frac{\sec^2 3x}{4-\tan 3x}\,dx = -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|4-\tan 3x| + C$.

82. Let $u = 1 + \sin^2 x$. Then $du = 2\sin x\cos x\,dx = \sin 2x\,dx$, so
$\int \frac{\sin 2x}{1+\sin^2 x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln\left(1+\sin^2 x\right) + C$.

84. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $I = \int \frac{\ln x\sqrt{1+\ln x}}{x}\,dx = \int u(1+u)^{1/2}\,du$. Now let $v = 1 + u$. Then $dv = du$ and
$I = \int (v-1)v^{1/2}\,dv = \int \left(v^{3/2} - v^{1/2}\right)dv = \frac{2}{5}v^{5/2} - \frac{2}{3}v^{3/2} + C = \frac{2}{15}v^{3/2}(3v-5) + C$
$= \frac{2}{15}(1+u)^{3/2}(3u-2) + C = \frac{2}{15}(\ln x + 1)^{3/2}(3\ln x - 2) + C$

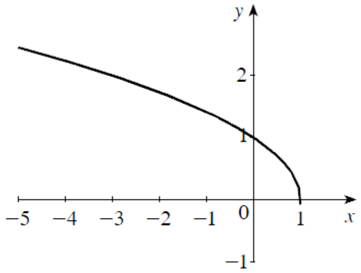
91. $\frac{dy}{dx} = \frac{d}{dx}\int_x^{x^2} \ln t\,dt = \frac{d}{dx}\left[\int_x^c \ln t\,dt + \int_c^{x^2} \ln t\,dt\right] = \frac{d}{dx}\left[-\int_c^x \ln t\,dt + \int_c^{x^2} \ln t\,dt\right] = -\ln x + \left(\ln x^2\right)\frac{d}{dx}\left(x^2\right)$
$= -\ln x + 2x\ln x^2 = (4x-1)\ln x$

92. **a.** $y = \int_{2/x}^{x^2} \frac{dt}{t} = \ln|t|\Big|_{2/x}^{x^2} = \ln x^2 - \ln\frac{2}{x} = 2\ln x - \ln 2 + \ln x = 3\ln x - \ln 2$, so $\frac{dy}{dx} = \frac{d}{dx}(3\ln x - \ln 2) = \frac{3}{x}$.

b. $\frac{dy}{dx} = \frac{d}{dx}\int_{2/x}^{x^2} \frac{dt}{t} = \frac{d}{dx}\left[\int_{2/x}^{c} \frac{dt}{t} + \int_c^{x^2} \frac{dt}{t}\right] = \frac{d}{dx}\left[-\int_c^{2/x} \frac{dt}{t} + \int_c^{x^2} \frac{dt}{t}\right] = -\frac{1}{2/x}\frac{d}{dx}\left(\frac{2}{x}\right) + \frac{1}{x^2}\frac{d}{dx}\left(x^2\right)$
$= \left(-\frac{x}{2}\right)\left(-\frac{2}{x^2}\right) + \frac{1}{x^2}(2x) = \frac{1}{x} + \frac{2}{x} = \frac{3}{x}$

6.2 Inverse Functions

15. $f(x) = \sqrt{1-x}$ is one-to-one. There is no horizontal line that cuts the graph of f at more than one point.



20. By definition, $f\left(f^{-1}(7)\right) = 7$.

22. By inspection $f(0) = 2$, so $f^{-1}(2) = 0$.

26. By inspection $f(1) = 1$, so $f^{-1}(1) = 1$.

50. **a.** $f(x) = \dfrac{x+1}{2x-1} \Rightarrow f(1) = \dfrac{2}{1} = 2$, so $(1, 2)$ lies on the graph of f.

b. $f'(x) = \dfrac{(2x-1)-(x+1)(2)}{(2x-1)^2} = -\dfrac{3}{(2x-1)^2} \Rightarrow g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(1)} = -\left.\dfrac{(2x-1)^2}{3}\right|_{x=1} = -\dfrac{1}{3}$

53. **a.** $f(x) = \dfrac{1}{1+x^2} \Rightarrow f(2) = \frac{1}{5}$, so $\left(2, \frac{1}{5}\right)$ lies on the graph of f.

b. $f'(x) = -\dfrac{2x}{(1+x^2)^2} \Rightarrow g'\left(\frac{1}{5}\right) = \dfrac{1}{f'\left(g\left(\frac{1}{5}\right)\right)} = \dfrac{1}{f'(2)} = -\left.\dfrac{\left(1+x^2\right)^2}{2x}\right|_{x=2} = -\dfrac{25}{4}$

56. $H'(x) = \dfrac{d}{dx}\left[g(g(x))\right] = g'(g(x))\,g'(x)$, so

$H'(3) = g'(g(3))\,g'(3) = g'(4)\,g'(3) = \dfrac{1}{f'(g(4))} \cdot \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(5)} \cdot \dfrac{1}{f'(4)} = \dfrac{1}{2} \cdot \dfrac{1}{\frac{1}{2}} = 1.$

57. Observe that $f(2) = \displaystyle\int_2^2 \frac{dt}{\sqrt{1+t^3}} = 0$, showing that $(2, 0)$ lies on the graph of f. Next, observe that $f'(x) = \dfrac{d}{dx}\displaystyle\int_2^x \frac{dt}{\sqrt{1+t^3}} = \frac{1}{\sqrt{1+x^3}}$ (by the FTC, Part 1). Therefore, $\left(f^{-1}\right)'(0) = \dfrac{1}{f'\left(f^{-1}(0)\right)} = \dfrac{1}{f'(2)} = \sqrt{1+2^3} = 3.$