3.5 Concept Questions

1. a. $\lim_{x\to 3} f(x) = \infty$ means $f(x)$ can be made as large as we please by taking x sufficiently close to (but not equal to) 3.

b. $\lim_{x\to 2^-} f(x) = -\infty$ means $f(x)$ can be made as large in absolute value as we please (but negative) by taking x sufficiently close to (but less than) 2.

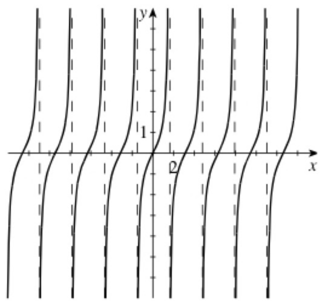
2. a. $\lim_{x\to -\infty} f(x) = 2$ means $f(x)$ can be made as close to 2 as we please by taking x sufficiently large in absolute value and negative.

b. $\lim_{x\to \infty} f(x) = -5$ means $f(x)$ can be made as close to -5 as we please by taking x sufficiently large.

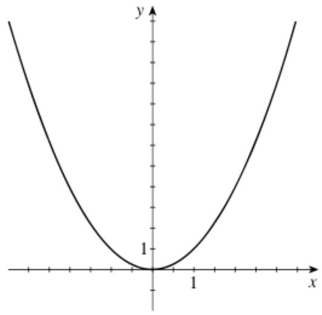
3. a. See page 292.

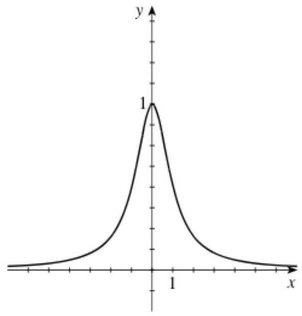
b. See page 296.

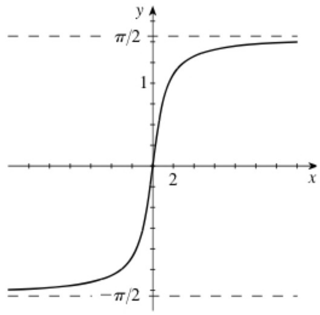
4. a. The graph of a function can have infinitely many vertical asymptotes. For example, $f(x) = \tan x$ has vertical asymptotes at $x = \frac{\pi}{2} \pm n\pi$, $n = 0, 1, 2, 3, \ldots$.



b. The graph of a function can have zero, one, or two horizontal asymptotes. For example, $f(x) = x^2$ has no asymptote, $f(x) = \dfrac{1}{x^2+1}$ has $y = 0$ as its only horizontal asymptote, and $f(x) = \tan^{-1} x$ has horizontal asymptotes at $y = \pm\frac{\pi}{2}$.







3.5 Limits Involving Infinity and Asymptotes

1. a. $\lim_{x\to 0^-} f(x) = -\infty$ **b.** $\lim_{x\to 0^+} f(x) = \infty$ **c.** $\lim_{x\to \infty} f(x) = \infty$ **d.** $\lim_{x\to -\infty} f(x) = -\infty$

2. a. $\lim_{x\to 0^-} f(x) = \infty$ **b.** $\lim_{x\to 0^+} f(x) = -\infty$ **c.** $\lim_{x\to \infty} f(x) = \infty$ **d.** $\lim_{x\to -\infty} f(x) = \infty$

3. a. $\lim_{x\to 0} f(x) = -\infty$ **b.** $\lim_{x\to -\infty} f(x) = 0$ **c.** $\lim_{x\to \infty} f(x) = 0$

4. a. $\lim_{x\to -\infty} f(x) = 1$ **b.** $\lim_{x\to \infty} f(x) = 1$

5. $\lim_{x\to 2n\pi} f(x) = \infty$

6. a. $\lim_{x\to -\infty} f(x)$ does not exist. **b.** $\lim_{x\to \infty} f(x)$ does not exist.

8. $\lim_{t\to -3^+} \dfrac{t}{t+3} = -\infty$ since the numerator approaches -3 and the denominator approaches 0 through positive values as $t \to -3$ from the right.

10. $\lim_{x\to 1^+} \dfrac{x+1}{1-x} = -\infty$ since the numerator approaches 2 and the denominator approaches 0 through negative values as $x \to 1$ from the right.

14. $\lim_{x\to -1^+} \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) = -\infty$. As $x \to -1$ from the right, the first term approaches -1 but the second term approaches ∞.

16. $\lim_{x\to 0^+} \frac{1}{\sin x} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 0$ from the right.

20. $\lim_{x\to\infty} \frac{x+1}{x-5} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1-\frac{5}{x}} = 1$

22. $\lim_{x\to\infty} \frac{2x^2-1}{4x^2+1} = \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{4+\frac{1}{x^2}} = \frac{1}{2}$

26. $\lim_{x\to-\infty} \frac{x^4+1}{x^3+1} = \lim_{x\to-\infty} \frac{x+\frac{1}{x^3}}{1+\frac{1}{x^3}} = -\infty$

28. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)\left(\frac{x^2+1}{x^2-1}\right) = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)\left(\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}\right) = 1$

32. $\lim_{t\to-\infty} \frac{2t^2}{\sqrt{t^4+t^2}} = \lim_{t\to-\infty} \frac{2t^2}{\sqrt{t^4+t^2}} \cdot \frac{\frac{1}{t^2}}{\frac{1}{t^2}} = \lim_{t\to-\infty} \frac{2}{\sqrt{1+\frac{1}{t^2}}} = 2$

34. $\lim_{x\to\infty} \cos\frac{1}{x} = 1$ because as $x \to \infty$, $\frac{1}{x} \to 0$ and $\cos 0 = 1$.

36. $\lim_{x\to\infty} \frac{x}{3x+\cos x} = \lim_{x\to\infty} \frac{1}{3+\frac{\cos x}{x}} = \frac{1}{3}$

50. $\lim_{x\to\infty} \frac{x}{x+1} = 1$, so $y = 1$ is a horizontal asymptote. $\lim_{x\to-1^+} \frac{x}{x+1} = -\infty$, so $x = -1$ is a vertical asymptote.

52. $\lim_{t\to\infty} \frac{t^2}{t^2-4} = 1$, so $y = 1$ is a horizontal asymptote. $\lim_{t\to-2^-} \frac{t^2}{(t+2)(t-2)} = \infty$ and $\lim_{t\to 2^-} \frac{t^2}{(t+2)(t-2)} = -\infty$ so $t = \pm 2$ are vertical asymptotes.

54. $\lim_{x\to\infty} \frac{2-x^2}{x^2+x} = -1$, so $y = -1$ is a horizontal asymptote. $\lim_{x\to-1^-} \frac{2-x^2}{x(x+1)} = \infty$ and $\lim_{x\to 0^+} \frac{2-x^2}{x(x+1)} = \infty$, so $x = -1$ and $x = 0$ are vertical asymptotes.

56. If $x > 0$, then $\sqrt{x^6} = x^3$. Dividing the numerator and the denominator of $f(x)$ by x^3, we have

$$f(x) = \frac{2x^3}{\sqrt{3x^6+2}} = \frac{2}{\frac{1}{x^3}\sqrt{3x^6+2}} = \frac{2}{\frac{1}{\sqrt{x^6}}\sqrt{3x^6+2}} = \frac{2}{\sqrt{\frac{1}{x^6}(3x^6+2)}} = \frac{2}{\sqrt{3+\frac{2}{x^6}}}, \text{ so}$$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2}{\sqrt{3+\frac{2}{x^6}}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. We conclude that $y = \frac{2\sqrt{3}}{3}$ is a horizontal asymptote of the graph of f.

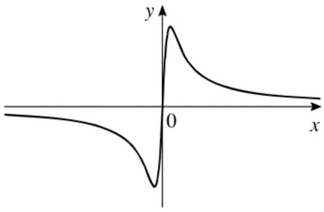
Next, if $x < 0$, then $\sqrt{x^6} = \left|x^3\right| = -x^3$. Dividing the numerator and the denominator of $f(x)$ by $-x^3$, we have $f(x) = \frac{2}{\frac{1}{x^3}\sqrt{3x^6+2}} = \frac{2}{-\frac{1}{\sqrt{x^6}}\sqrt{3x^6+2}} = -\frac{2}{\sqrt{\frac{1}{x^6}(3x^6+2)}} = -\frac{2}{\sqrt{3+\frac{2}{x^6}}}$, so

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-2}{\sqrt{3+\frac{2}{x^6}}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$. We see that $y = -\frac{2\sqrt{3}}{3}$ is also a horizontal asymptote of the graph of f.

83. False. $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$ and $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$, so $\lim_{x \to 2} \frac{1}{x-2}$ does not exist.

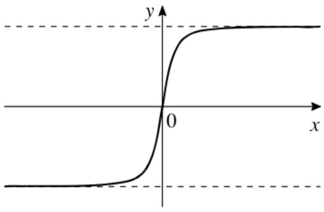
84. True. Write $f(x) = c$ and let $\varepsilon > 0$ be given. Then pick N to be any positive number. Then $x > N \Rightarrow$ $|f(x) - c| = |c - c| < \varepsilon$.

85. False. The graph of $f(x) = \frac{2x}{x^2+1}$ crosses its horizontal asymptote $y = 0$.



86. False. Let $f(x) = \frac{x^2-4}{x-2}$. Then $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = 4$, and so $x = 2$ is not a vertical asymptote.

87. True. Let $f(x) = \frac{x}{\sqrt{x^2+1}}$. Then $y = -1$ and $y = 1$ are horizontal asymptotes.



88. False. Let $f(x) = x + 2$. Then $\lim_{x \to 0^+} f(x) = 2 = L$, but $\lim_{x \to \infty} f\left(\frac{1}{x}\right) = \lim_{x \to \infty} \left(\frac{1}{x} + 2\right) = 2 \neq \frac{1}{2} = \frac{1}{L}$.