9.1

13. $\lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = 2$

28. $\lim_{n\to\infty} \sin \frac{n\pi}{2n+1} = \sin\left(\lim_{n\to\infty} \frac{n\pi}{2n+1}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{2+\frac{1}{n}}\right) = \sin\frac{\pi}{2} = 1$

32. $\lim_{n\to\infty} \frac{\ln n^2}{\sqrt{n}} = 0$. By l'Hôpital's Rule, $\lim_{x\to\infty} \frac{\ln x^2}{\sqrt{x}} = 2\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = 2\lim_{x\to\infty} \frac{1/x}{1/\left(2x^{1/2}\right)} = \lim_{x\to\infty} \frac{4}{x^{1/2}} = 0.$

39. $0 < \frac{\sin^2 n}{\sqrt{n}} < \frac{1}{\sqrt{n}}$. Because $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the Squeeze Theorem shows that $\lim_{n\to\infty} \frac{\sin^2 n}{\sqrt{n}} = 0$.

73. a. $a_n = \frac{a_{n-1}}{2+a_{n-1}} < a_{n-1}$ for $n \geq 1$, so $\{a_n\}$ is decreasing. Now because all the terms are positive, $\{a_n\}$ is bounded below by 0. By the Monotone Convergence Theorem for Sequences, the sequence converges.

b. Suppose $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} a_{n-1} = L$ as well, so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_{n-1}}{2+a_{n-1}} = \frac{\lim_{n\to\infty} a_{n-1}}{\lim_{n\to\infty}\left(2+a_{n-1}\right)} = \frac{L}{2+L}$

$\Rightarrow \frac{L}{2+L} = L \Rightarrow L = 2L + L^2 \Rightarrow L\left(L+1\right) = 0 \Rightarrow L = 0$ or $L = -1$. Because $\{a_n\}$ is bounded below by 0, we see that $L = 0$; that is, $\lim_{n\to\infty} a_n = 0$.

9. 2

3. $S_n = \sum_{k=1}^{n} \frac{4}{(2k+3)(2k+5)} = \sum_{k=1}^{n} \left(\frac{2}{2k+3} - \frac{2}{2k+5} \right)$

$= \left(\frac{2}{5} - \frac{2}{7}\right) + \left(\frac{2}{7} - \frac{2}{9}\right) + \cdots + \left(\frac{2}{2n+3} - \frac{2}{2n+5}\right) = \frac{2}{5} - \frac{2}{2n+5}$

so $\sum_{n=1}^{\infty} \frac{4}{(2n+3)(2n+5)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{2}{5} - \frac{2}{2n+5}\right) = \frac{2}{5}$.

9. $\frac{5}{3} - \frac{5}{9} + \frac{5}{27} - \frac{5}{81} + \cdots = \frac{5}{3} \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \frac{\frac{5}{3}}{1 - \left(-\frac{1}{3}\right)} = \frac{5}{4}$

17. Since $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$, the series diverges.

42. $S_n = \sum_{k=1}^{n} \ln \frac{k}{k+1} = \sum_{k=1}^{n} [\ln k - \ln(k+1)] = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + [\ln n - \ln(n+1)] = -\ln(n+1)$.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [-\ln(n+1)] = -\infty$, so $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ diverges.

57. $1.\overline{213} = 1 + \frac{213}{1000} + \frac{213}{10^6} + \cdots = 1 + \frac{213}{1000}\left[1 + \left(\frac{1}{10^3}\right) + \left(\frac{1}{10^3}\right)^2 + \cdots\right] = 1 + \frac{213}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} = \frac{404}{333}$