11.1 Vectors in the Plane ET 10.1

27. $2\mathbf{a} = 2\langle -1, 2\rangle = \langle -2, 4\rangle$, $\mathbf{a} + \mathbf{b} = \langle -1, 2\rangle + \langle 3, 1\rangle = \langle 2, 3\rangle$, $\mathbf{a} - \mathbf{b} = \langle -1, 2\rangle - \langle 3, 1\rangle = \langle -4, 1\rangle$, $2\mathbf{a} + \mathbf{b} = 2\langle -1, 2\rangle + \langle 3, 1\rangle = \langle 1, 5\rangle$, and $|2\mathbf{a} + \mathbf{b}| = |\langle 1, 5\rangle| = \sqrt{1^2 + 5^2} = \sqrt{26}$.

29. $2\mathbf{a} = 2(3\mathbf{i} - 2\mathbf{j}) = 6\mathbf{i} - 4\mathbf{j}$, $\mathbf{a} + \mathbf{b} = (3\mathbf{i} - 2\mathbf{j}) + 2\mathbf{i} = 5\mathbf{i} - 2\mathbf{j}$, $\mathbf{a} - \mathbf{b} = (3\mathbf{i} - 2\mathbf{j}) - 2\mathbf{i} = \mathbf{i} - 2\mathbf{j}$, $2\mathbf{a} + \mathbf{b} = 2(3\mathbf{i} - 2\mathbf{j}) + 2\mathbf{i} = 8\mathbf{i} - 4\mathbf{j}$, and $|2\mathbf{a} + \mathbf{b}| = |8\mathbf{i} - 4\mathbf{j}| = \sqrt{8^2 + 4^2} = 4\sqrt{5}$.

31. $2\mathbf{a} = 2\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) = \mathbf{i} + 3\mathbf{j}$, $\mathbf{a} + \mathbf{b} = \left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) + \left(\frac{3}{4}\mathbf{i} - \frac{1}{4}\mathbf{j}\right) = \frac{5}{4}\mathbf{i} + \frac{5}{4}\mathbf{j}$, $\mathbf{a} - \mathbf{b} = \left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) - \left(\frac{3}{4}\mathbf{i} - \frac{1}{4}\mathbf{j}\right) = -\frac{1}{4}\mathbf{i} + \frac{7}{4}\mathbf{j}$, $2\mathbf{a} + \mathbf{b} = 2\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) + \left(\frac{3}{4}\mathbf{i} - \frac{1}{4}\mathbf{j}\right) = \frac{7}{4}\mathbf{i} + \frac{11}{4}\mathbf{j}$, and $|2\mathbf{a} + \mathbf{b}| = \left|\frac{7}{4}\mathbf{i} + \frac{11}{4}\mathbf{j}\right| = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{11}{4}\right)^2} = \frac{\sqrt{170}}{4}$.

33. $2\mathbf{a} - 3\mathbf{b} = 2(2\mathbf{i}) - 3(-6\mathbf{j}) = 4\mathbf{i} + 18\mathbf{j}$, $\frac{1}{2}\mathbf{a} + \frac{1}{3}\mathbf{b} = \frac{1}{2}(2\mathbf{i}) + \frac{1}{3}(-6\mathbf{j}) = \mathbf{i} - 2\mathbf{j}$

35. $a\mathbf{u} + b\mathbf{v} = \mathbf{w} \Rightarrow a\langle -1, 3\rangle + b\langle 2, 4\rangle = \langle 6, 4\rangle \Leftrightarrow \langle -a + 2b, 3a + 4b\rangle = \langle 6, 4\rangle \Leftrightarrow \left.\begin{array}{r} -a + 2b = 6 \\ 3a + 4b = 4 \end{array}\right\} \Leftrightarrow a = -1.6$ and $b = 2.2$.

49. A unit vector in the direction of $\mathbf{b}$ is $\mathbf{u} = \dfrac{\mathbf{b}}{|\mathbf{b}|} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$, and so $\mathbf{a} = 5\mathbf{u} = \left\langle \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right\rangle$.

53. $2\mathbf{a} - 3\mathbf{b} = 2\langle -3, 4\rangle - 3\langle 1, 2\rangle = \langle -9, 2\rangle$ and a unit vector in the direction of this vector is $\mathbf{u} = \dfrac{2\mathbf{a} - 3\mathbf{b}}{|2\mathbf{a} - 3\mathbf{b}|} = \dfrac{\langle -9, 2\rangle}{\sqrt{(-9)^2 + 2^2}} = \left\langle -\frac{9\sqrt{85}}{85}, \frac{2\sqrt{85}}{85}\right\rangle$. Thus, the required vector is $3\mathbf{u} = \left\langle -\frac{27\sqrt{85}}{85}, \frac{6\sqrt{85}}{85}\right\rangle$.

11.2 Coordinate Systems and Vectors in Three-Space ET 10.2

33. Denote the point on the sphere by $A(1, 3, 5)$. Then its radius is $d(A, C)$, where $C(-1, 2, 4)$ is its center. Thus, $r = \sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$. Therefore, the required equation is $(x+1)^2 + (y-2)^2 + (z-4)^2 = 6$.

37. $x^2 + y^2 + z^2 - 4x + 6y = 0$. Completing the squares in x and y, we obtain $\left[x^2 - 4x + (-2)^2\right] + \left[y^2 + 6y + (3)^2\right] + z^2 = 4 + 9$ $\Leftrightarrow (x-2)^2 + (y+3)^2 + z^2 = 13$. Thus, the sphere has center $(2, -3, 0)$ and radius $\sqrt{13}$.

53. $\mathbf{a} = \langle -1, 2, 0\rangle$ and $\mathbf{b} = \langle 2, 3, -1\rangle$, so $\mathbf{a} + \mathbf{b} = \langle 1, 5, -1\rangle$,
$2\mathbf{a} - 3\mathbf{b} = 2\langle -1, 2, 0\rangle - 3\langle 2, 3, -1\rangle = \langle -2, 4, 0\rangle - \langle 6, 9, -3\rangle = \langle -8, -5, 3\rangle$,
$|3\mathbf{a}| = |3\langle -1, 2, 0\rangle| = |\langle -3, 6, 0\rangle| = \sqrt{(-3)^2 + 6^2 + 0^2} = 3\sqrt{5}$,
$|-2\mathbf{b}| = |-2\langle 2, 3, -1\rangle| = |\langle -4, -6, 2\rangle| = \sqrt{(-4)^2 + (-6)^2 + 2^2} = 2\sqrt{14}$, and
$|\mathbf{a} - \mathbf{b}| = |\langle -1, 2, 0\rangle - \langle 2, 3, -1\rangle| = |\langle -3, -1, 1\rangle| = \sqrt{(-3)^2 + (-1)^2 + 1^2} = \sqrt{11}$.

56. $\mathbf{a} = -2\mathbf{i} + 4\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$, so $\mathbf{a} + \mathbf{b} = -2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,

$2\mathbf{a} - 3\mathbf{b} = 2(-2\mathbf{i} + 4\mathbf{k}) - 3(2\mathbf{j} - \mathbf{k}) = (-4\mathbf{i} + 8\mathbf{k}) - (6\mathbf{j} - 3\mathbf{k}) = -4\mathbf{i} - 6\mathbf{j} + 11\mathbf{k}$,

$|3\mathbf{a}| = |3(-2\mathbf{i} + 4\mathbf{k})| = |-6\mathbf{i} + 12\mathbf{k}| = \sqrt{(-6)^2 + 12^2} = 6\sqrt{5}$,

$|-2\mathbf{b}| = |-2(2\mathbf{j} - \mathbf{k})| = |-4\mathbf{j} + 2\mathbf{k}| = \sqrt{(-4)^2 + 2^2} = 2\sqrt{5}$, and

$|\mathbf{a} - \mathbf{b}| = |(-2\mathbf{i} + 4\mathbf{k}) - (2\mathbf{j} - \mathbf{k})| = |-2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}| = \sqrt{(-2)^2 + (-2)^2 + 5^2} = \sqrt{33}$.

63. $\mathbf{a} = \langle 1, 2, 2 \rangle$, so $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 2^2} = 3$.

a. $\mathbf{u}_1 = \frac{1}{3}\mathbf{a} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

b. $\mathbf{u}_2 = \left\langle -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \right\rangle$

65. $\mathbf{a} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, so $|\mathbf{a}| = \sqrt{(-1)^2 + 3^2 + (-1)^2} = \sqrt{11}$.

a. $\mathbf{u}_1 = \frac{1}{\sqrt{11}}(-\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = -\frac{\sqrt{11}}{11}\mathbf{i} + \frac{3\sqrt{11}}{11}\mathbf{j} - \frac{\sqrt{11}}{11}\mathbf{k}$

b. $\mathbf{u}_2 = \frac{\sqrt{11}}{11}\mathbf{i} - \frac{3\sqrt{11}}{11}\mathbf{j} + \frac{\sqrt{11}}{11}\mathbf{k}$

71. $\mathbf{a} = \langle 3, -1, 2 \rangle$ and $\mathbf{b} = \langle 1, 0, -1 \rangle$, so $\mathbf{a} - 2\mathbf{b} = \langle 3, -1, 2 \rangle - 2\langle 1, 0, -1 \rangle = \langle 1, -1, 4 \rangle$ and $|\mathbf{a} - 2\mathbf{b}| = \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{18} = 3\sqrt{2}$. Therefore, a unit vector in the direction of $\mathbf{a} - 2\mathbf{b}$ is $\mathbf{u} = \frac{1}{3\sqrt{2}}\langle 1, -1, 4 \rangle$, and the required vector is $2\mathbf{u} = \frac{2}{3\sqrt{2}}\langle 1, -1, 4 \rangle = \left\langle \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}, \frac{4\sqrt{2}}{3} \right\rangle$.

11.3 The Dot Product ET 10.3

5. $\mathbf{a} \cdot \mathbf{b} = (2\mathbf{i} + 3\mathbf{j}) \cdot (\mathbf{i} - 2\mathbf{j}) = 2 - 6 = -4$

7. $\mathbf{a} \cdot \mathbf{b} = \langle 0, 1, -3 \rangle \cdot \langle 10, \pi, -\pi \rangle = \pi + 3\pi = 4\pi$

23. $\cos 45^\circ = \dfrac{\langle 1, c \rangle \cdot \langle 1, 2 \rangle}{\sqrt{1 + c^2}\sqrt{1 + 4^2}} \Rightarrow \dfrac{\sqrt{2}}{2} = \dfrac{1 + 2c}{\sqrt{1 + c^2}\sqrt{5}} \Rightarrow \sqrt{10}\sqrt{1 + c^2} = 2 + 4c \Rightarrow 10\left(1 + c^2\right) = 4 + 16c + 16c^2 \Rightarrow$ $6c^2 + 16c - 6 = 0 \Rightarrow (3c - 1)(c + 3) = 0 \Rightarrow c = -3$ or $\frac{1}{3}$. If $c = \frac{1}{3}$, then $\theta = 45^\circ$, and if $c = -3$, then $\theta = 135^\circ$, so the desired value of c is $\frac{1}{3}$.

31. If $\langle c, 2, -1 \rangle$ and $\langle 2, 3, c \rangle$ are orthogonal, then $\langle c, 2, -1 \rangle \cdot \langle 2, 3, c \rangle = 0$; that is, $2c + 6 - c = 0 \Leftrightarrow c = -6$.

32. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a vector orthogonal to both $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{k}$. Then

$$\left.\begin{aligned} \mathbf{v} \cdot \mathbf{a} &= a + b + c = 0 \\ \mathbf{v} \cdot \mathbf{b} &= -2a \quad\;\; + c = 0 \end{aligned}\right\}$$

Solving the system, we find $a = \frac{1}{2}c$ and $b = -\frac{3}{2}c$, so $\mathbf{v} = \frac{1}{2}c\mathbf{i} - \frac{3}{2}c\mathbf{j} + c\mathbf{k} \Rightarrow$ $|\mathbf{v}| = \sqrt{\frac{1}{4}c^2 + \frac{9}{4}c^2 + c^2} = \frac{\sqrt{14}}{2}|c|$. Thus, one such unit vector is $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \frac{\sqrt{14}}{14}\mathbf{i} - \frac{3\sqrt{14}}{14}\mathbf{j} + \frac{\sqrt{14}}{7}\mathbf{k}$. Note that the other unit vector is $-\dfrac{\mathbf{v}}{|\mathbf{v}|} = -\frac{\sqrt{14}}{14}\mathbf{i} + \frac{3\sqrt{14}}{14}\mathbf{j} - \frac{\sqrt{14}}{7}\mathbf{k}$.

37. $\alpha = \frac{\pi}{3}$ and $\gamma = \frac{\pi}{4}$. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have $\cos^2\left(\frac{\pi}{3}\right) + \cos^2\beta + \cos^2\left(\frac{\pi}{4}\right) = 1 \Rightarrow$ $\cos^2\beta = 1 - \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{4}$. Thus, $\cos\beta = \pm\frac{1}{2}$, and so $\beta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

41. $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{k}$.

a. $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \left[\dfrac{(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\cdot(3\mathbf{i}+\mathbf{k})}{4+1+16}\right](2\mathbf{i}+\mathbf{j}+4\mathbf{k}) = \frac{6+4}{21}\,(2\mathbf{i}+\mathbf{j}+4\mathbf{k}) = \frac{20}{21}\mathbf{i} + \frac{10}{21}\mathbf{j} + \frac{40}{21}\mathbf{k}$

b. $\text{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left[\dfrac{(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\cdot(3\mathbf{i}+\mathbf{k})}{9+1}\right](3\mathbf{i}+\mathbf{k}) = \frac{6+4}{10}\,(3\mathbf{i}+\mathbf{k}) = 3\mathbf{i}+\mathbf{k}$

43. $\mathbf{a} = \langle -3, 4, -2\rangle$ and $\mathbf{b} = \langle 0, 1, 0\rangle$.

a. $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \left(\dfrac{\langle -3, 4, -2\rangle\cdot\langle 0, 1, 0\rangle}{9+16+4}\right)\langle -3, 4, -2\rangle = \frac{4}{29}\,\langle -3, 4, -2\rangle = \left\langle -\frac{12}{29}, \frac{16}{29}, -\frac{8}{29}\right\rangle$

b. $\text{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left(\dfrac{\langle -3, 4, -2\rangle\cdot\langle 0, 1, 0\rangle}{1}\right)\langle 0, 1, 0\rangle = 4\,\langle 0, 1, 0\rangle = \langle 0, 4, 0\rangle$