2.1 The Derivative

12. $f(x) = -\dfrac{2}{\sqrt{x}} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-\dfrac{2}{\sqrt{x+h}}-\left(-\dfrac{2}{\sqrt{x}}\right)}{h} = 2\lim_{h\to 0}\frac{\dfrac{-\sqrt{x}+\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= 2\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = 2\lim_{h\to 0}\frac{(x+h)-x}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= 2\lim_{h\to 0}\frac{h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h}+\sqrt{x}\right)} = 2\lim_{h\to 0}\frac{1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{2}{\sqrt{x}\sqrt{x}\left(2\sqrt{x}\right)}$$

$= \dfrac{1}{x\sqrt{x}}$ with domain $(0, \infty)$.

25. $y = f(x) = \sqrt{2x} \quad \Rightarrow$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} = \lim_{h\to 0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\cdot\frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}}$$

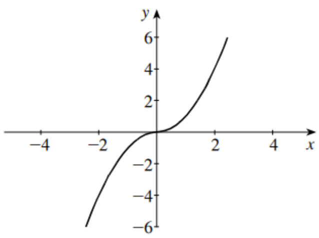
$$= \lim_{h\to 0}\frac{2(x+h)-2x}{h\left[\sqrt{2(x+h)}+\sqrt{2x}\right]} = \lim_{h\to 0}\frac{2}{\sqrt{2(x+h)}+\sqrt{2x}} = \frac{1}{\sqrt{2x}}$$

$\left.\dfrac{dy}{dx}\right|_{x=2} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$, so y is increasing at the rate of $\frac{1}{2}$ unit per unit change in x.

50. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+1) = 1$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2+1\right) = 1$. Therefore, $\lim_{x\to 0} f(x) = 1$. Also, $f(0) = 0+1 = 1$, and so $\lim_{x\to 0} f(x) = f(0)$. Therefore, f is continuous at 0.

To show that f is not differentiable at 0, let $h < 0$ and consider $\lim_{h\to 0^-}\dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\dfrac{(h+1)-1}{h} = \lim_{h\to 0^-} 1 = 1$. Next, if $h > 0$, then $\lim_{h\to 0^+}\dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\dfrac{\left[(0+h)^2+1\right]-1}{h} = \lim_{h\to 0^+} h = 0$. This shows that $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist, and so by definition, f is not differentiable at 0.

58. a.



$$f(x) = x|x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

b. If $x \geq 0$, then $f(x) = x^2$, so

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{\left(x^2+2xh+h^2\right)-x^2}{h} = \lim_{h\to 0}(2x+h) = 2x$$

If $x < 0$, then $f(x) = -x^2$, and a similar calculation shows that $f'(x) = -2x$. So f is differentiable everywhere.

c. From the results of part b, we see that $f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$

2.2 Basic Rules of Differentiation

19. $f(x) = \dfrac{x^3-4x^2+3}{x} = x^2-4x+3x^{-1} \Rightarrow f'(x) = \dfrac{d}{dx}\left(x^2-4x+3x^{-1}\right) = 2x-4-3x^{-2} = 2x-4-\dfrac{3}{x^2}$

40. $g'(x) = x^2-x-1 = -1 \Rightarrow x^2-x = x(x-1) = 0 \Rightarrow x = 0$ or 1. $g(0) = 1$ and $g(1) = \frac{1}{3}-\frac{1}{2}-1+1 = -\frac{1}{6}$, so the points are $(0, 1)$ and $\left(1, -\frac{1}{6}\right)$.

48. $y = \frac{1}{3}x^3 - 2x + 5 \Rightarrow \dfrac{dy}{dx} = x^2 - 2$. The slope of the given line is 2, so set $x^2 - 2 = 2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. The required points are $\left(-2, \frac{19}{3}\right)$ and $\left(2, \frac{11}{3}\right)$.

49. $y = \frac{1}{3}x^3 - 2x + 5 \Rightarrow \dfrac{dy}{dx} = x^2 - 2$. The slope of the given line is 1, so the normal line has slope -1. We set $x^2 - 2 = -1$ $\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$. The required points are $\left(-1, \frac{20}{3}\right)$ and $\left(1, \frac{10}{3}\right)$.