1.4 Continuous Functions

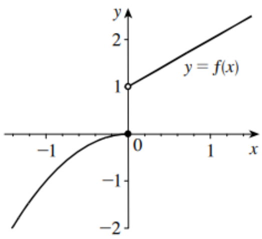
32. Since $\lim_{x\to 0^-}(x\cot kx) = \lim_{x\to 0^-}\left(\frac{x}{\sin kx}\cdot\frac{\cos kx}{1}\right) = \lim_{x\to 0^-}\left(\frac{1}{k}\cdot\frac{kx}{\sin kx}\right)\cdot\lim_{x\to 0^-}(\cos kx) = \frac{1}{k}$, we define $f(0) = 1/k$. Then $0 + c = 1/k$, or $c = 1/k$.

52. $\lim_{x\to 1}\frac{2x^3+x-3}{x-1} = \lim_{x\to 1}\frac{(x-1)\left(2x^2+2x+3\right)}{x-1} = \lim_{x\to 1}\left(2x^2+2x+3\right) = 7$. So f will be continuous at 1 if we define

$$f(x) = \begin{cases} \frac{2x^3+x-3}{x-1} & \text{if } x \neq 1 \\ 7 & \text{if } x = 1 \end{cases}$$

64. $f(x) = x^4 - 2x^3 - 3x^2 + 7$ is continuous on $[1, 2]$. $f(1) = 3 > 0$ and $f(2) = -5 < 0$. Therefore, by Theorem 7, $f(x) = 0$ has at least one root in $(1, 2)$.

70. **a.** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(-x^2\right) = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 1$. This shows that f is not continuous at 0, and therefore is not continuous on $[-1, 1]$.



b. $f(-1) = -1$ and $f(1) = 2$. The number $\frac{1}{2}$ lies between $f(-1)$ and $f(1)$, but there is no c in $[-1, 1]$ such that $f(c) = \frac{1}{2}$. (See the figure.)

85. Let $f(x) = a_{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0 = a_{2n+1}x^{2n+1}\left(1 + \frac{a_{2n}}{a_{2n+1}x} + \cdots + \frac{a_0}{a_{2n+1}x^{2n+1}}\right)$, where $a_{2n+1} \neq 0$ and $x \neq 0$. Without loss of generality, let us assume that $a_{2n+1} > 0$. Observe that f is continuous, $f(x) < 0$ if x is negative and sufficiently large in absolute value, and $f(x) > 0$ if x is positive and sufficiently large in absolute value. Therefore, we can find numbers a and b with $a < b$ such that $f(a) < 0$ and $f(b) > 0$. Using the Intermediate Value Theorem, we conclude that there exists at least one number c in (a, b) such that $f(c) = 0$. Thus, the given equation has at least one real root.

1.5 Tangent Lines and Rates of Change

16.
$$\lim_{h\to 0}\frac{g(-1+h)-g(-1)}{(-1+h)-(-1)} = \lim_{h\to 0}\frac{\left[(-1+h)^2-(-1+h)+2\right]-\left[(-1)^2-(-1)+2\right]}{h}$$
$$= \lim_{h\to 0}\frac{h^2-2h+1-h+1+2-4}{h} = \lim_{h\to 0}\frac{h(h-3)}{h} = -3$$

18. $\lim_{h\to 0}\frac{f(4+h)-f(4)}{(4+h)-4} = \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h} = \lim_{h\to 0}\frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} = \lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)} = \frac{1}{4}$

39. Using the definition of the derivative with $h = x - 1$, we find $f(x) = x^4$ and $a = 1$.