9.7 Power Series

5. Let $u_n = \frac{(2x)^n}{n!}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)!}\cdot\frac{n!}{(2x)^n}\right| = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)|x| = 0$ for any real x, so the radius of convergence is infinite and the interval of convergence is $(-\infty,\infty)$.

8. Let $u_n = \frac{n!x^n}{(2n)!}$. Then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right| = \lim_{n\to\infty}\frac{|x|}{2(2n+1)} = 0$ for any real x, so the radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$.

21. Let $u_n = \frac{2^n(x+2)^n}{n^n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x+2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n(x+2)^n}\right| = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)\left(\frac{n}{n+1}\right)^n|x+2| = 0$$ for any real x since

$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = \frac{1}{e}$. Thus, $R=\infty$ and the interval of convergence is $(-\infty,\infty)$.

25. Let $u_n = \frac{x^n}{n(\ln n)^2}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)[\ln(n+1)]^2}\cdot\frac{n(\ln n)^2}{x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left[\frac{\ln n}{\ln(n+1)}\right]^2|x| = |x|,$$ so $R=1$ and the series converges on $(-1,1)$. At $x=-1$ the series is $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$, which converges, and at $x=1$ it is $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$, which also converges (see Exercise 9.3.33). Thus, the interval of convergence is $[-1,1]$.

9.8 Taylor and Maclaurin Series

8. $f(x)=\frac{1}{x},\quad f'(x)=-\frac{1}{x^2},\ f''(x)=\frac{2}{x^3},\quad f'''(x)=-\frac{3\cdot 2}{x^4},\ \ldots,\ f^{(n)}(x)=\frac{(-1)^n n!}{x^{n+1}},\ \ldots$

$f(-1)=-1,\ f'(-1)=-1,\ f''(-1)=-2,\ f'''(-1)=-3\cdot 2,\ \ldots,\ f^{(n)}(-1)=-n!,\quad \ldots$

The required Taylor series is $\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n = \sum_{n=0}^{\infty}\frac{-n!}{n!}(x+1)^n = -\sum_{n=0}^{\infty}(x+1)^n$.

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(x+1)^n}\right| = |x+1|$, so $R=1$.

16. $f(x)=\frac{1}{4+x^2}=\frac{1}{4}\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}=\frac{1}{4}\cdot\frac{1}{1-\left[-\left(\frac{x}{2}\right)^2\right]}=\frac{1}{4}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{2}\right)^2\right]^n=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}x^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n}$. The series converges if $\left|\frac{x}{2}\right|<1$ or $|x|<2$, so $R=2$.

21. $f(x)=\cos^2 x=\frac{1}{2}(1+\cos 2x)=\frac{1}{2}+\frac{1}{2}\cos 2x=\frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}=1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!}$ with $R=\infty$.

33. $f(x)=(1-x)^{3/5}=[1+(-x)]^{3/5}=1+\frac{3}{5}(-x)+\frac{\frac{3}{5}\left(\frac{3}{5}-1\right)}{2!}x^2+\cdots+\frac{\frac{3}{5}\left(\frac{3}{5}-1\right)\left(\frac{3}{5}-2\right)\cdots\left(\frac{3}{5}-n+1\right)}{n!}x^n+\cdots$

$=1-\frac{3}{5}x-3\sum_{n=2}^{\infty}\frac{2\cdot 7\cdot 12\cdot\cdots\cdot(5n-8)}{n!5^n}x^n$ with $R=1$.