9.1 Sequences

18. $\lim_{n\to\infty} \frac{n^2-1}{2n^2+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2}$

22. $\lim_{n\to\infty} \frac{\sqrt{2n^2+1}}{n} = \lim_{n\to\infty} \frac{\sqrt{2+\frac{1}{n^2}}}{1} = \sqrt{2}$

24. $\lim_{n\to\infty} \left[1+\left(-\frac{2}{e}\right)^n\right] = 1+0 = 1$

32. $\lim_{n\to\infty} \frac{\ln n^2}{\sqrt{n}} = 0$. By l'Hôpital's Rule, $\lim_{x\to\infty} \frac{\ln x^2}{\sqrt{x}} = 2\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = 2\lim_{x\to\infty} \frac{1/x}{1/\left(2x^{1/2}\right)} = \lim_{x\to\infty} \frac{4}{x^{1/2}} = 0.$

33. $\lim_{n\to\infty} \frac{2^n}{3^n+1} = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n}{1+\left(\frac{1}{3}\right)^n} = \frac{0}{1} = 0$

34. $\lim_{n\to\infty} \frac{2^n+1}{e^n} = \lim_{n\to\infty} \frac{\left(\frac{2}{e}\right)^n+\left(\frac{1}{e}\right)^n}{1} = \frac{0}{1} = 0$

36. $f(x) = \frac{x^p}{e^x}$ for $p > 0$. Using the result of Exercise 6.4.72, we see that $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$, so $\lim_{n\to\infty} \frac{n^p}{e^n} = 0$.

37. $\lim_{x\to\infty} \left[\left(1+\frac{2}{x}\right)^{1/x}\right] = \lim_{u\to\infty} \left[\left(1+\frac{1}{u}\right)^{1/(2u)}\right]$ (where $u = \frac{1}{2}x$) $= 1^0 = 1$

9.2 Series

18. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$, so the series diverges.

20. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{n-1}} = \sum_{n=0}^{\infty} 2\left(-\frac{3}{2}\right)^n$ is a divergent geometric series since $|r| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$.

29. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left[\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}\right]$ is a telescoping series.

$S_n = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)\right] = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$, so

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{3}{4}$ and so $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$.

32. $\displaystyle\sum_{n=0}^{\infty}\frac{3^{n+1}}{5^n}=3\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n=\frac{3}{1-\frac{3}{5}}=\frac{15}{2}$

39. $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{2^n}-\frac{1}{n\,(n+1)}\right]=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=1}^{\infty}\frac{1}{n\,(n+1)}$. Now $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\,(n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1$ by Example 9.2.1b, so $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{2^n}-\frac{1}{n\,(n+1)}\right]=1-1=0.$

42. $\displaystyle S_n=\sum_{k=1}^{n}\ln\frac{k}{k+1}=\sum_{k=1}^{n}\left[\ln k-\ln\,(k+1)\right]=(\ln 1-\ln 2)+(\ln 2-\ln 3)+\cdots+\left[\ln n-\ln\,(n+1)\right]=-\ln\,(n+1).$

$\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[-\ln\,(n+1)\right]=-\infty$, so $\displaystyle\sum_{n=1}^{\infty}\ln\frac{n}{n+1}$ diverges.

47. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+3^n}{6^n}=\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{1-\frac{1}{3}}+\frac{1}{1-\frac{1}{2}}=\frac{3}{2}+2=\frac{7}{2}$

49. $\displaystyle\lim_{n\to\infty}a_n=\lim_{n\to\infty}\tan^{-1}n=\frac{\pi}{2}\neq 0$, so by the Divergence Test, $\displaystyle\sum_{n=1}^{\infty}\tan^{-1}n$ diverges.

54. $\displaystyle\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=e^2\neq 0$, so $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^n$ diverges by the Divergence Test.

9.3 The Integral Test

16. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{2n^2+1}}$. Let $f\,(x)=\dfrac{x}{\sqrt{2x^2+1}}$. Then f is nonnegative, continuous, and decreasing on $[1,\infty)$.
$\displaystyle I=\int_1^{\infty}\frac{x\,dx}{\sqrt{2x^2+1}}=\lim_{b\to\infty}\int_1^b x\left(2x^2+1\right)^{-1/2}dx=\lim_{b\to\infty}\left(\tfrac{1}{2}\sqrt{2b^2+1}-\tfrac{1}{2}\sqrt{3}\right)=\infty$. Since I diverges, so does the series.

21. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$. Let $f\,(x)=\dfrac{\ln x}{x}$. Then f is nonnegative, continuous, and decreasing on $[2,\infty)$. But
$\displaystyle\int_2^{\infty}\frac{\ln x}{x}\,dx=\lim_{b\to\infty}\int_2^b\frac{\ln x}{x}\,dx=\lim_{b\to\infty}\left[\tfrac{1}{2}\,(\ln b)^2-\tfrac{1}{2}\,(\ln 2)^2\right]=\infty$, so the series diverges.

22. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$. Let $f\,(x)=\dfrac{\ln x}{x^2}$. Then f is nonnegative, continuous, and decreasing on $[2,\infty)$. We calculate $\displaystyle I=\int_2^{\infty}\frac{\ln x}{x^2}\,dx$ using the substitution $u=\ln x\Rightarrow du=\dfrac{dx}{x}$ and $x=e^u$. Then $\displaystyle I=\int\frac{u\,du}{e^u}=\int ue^{-u}\,du=-\,(1+u)\,e^{-u}+C$ (by parts).
Therefore, $\displaystyle I=\lim_{b\to\infty}\int_2^b\frac{\ln x}{x^2}\,dx=\lim_{b\to\infty}\left[-\frac{1+\ln x}{x}\right]_2^b=\lim_{b\to\infty}\left[-\frac{1+\ln b}{b}+\frac{1+\ln 2}{2}\right]=\frac{1+\ln 2}{2}$ by l'Hôpital's Rule.
Thus, the series converges.

23. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Let $f(x) = \frac{1}{x(\ln x)^2}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \left[-\frac{1}{\ln b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$, so the series converges.

24. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$. Let $f(x) = \frac{e^{1/x}}{x^2}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{e^{1/x}}{x^2}\, dx = \lim_{b\to\infty} \int_1^b \frac{e^{1/x}}{x^2}\, dx = \lim_{b\to\infty} \left(-e^{1/b} + e\right) = e - 1$, so the series converges.

30. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{e^{-n}+1} = 1 \neq 0$, so $\sum_{n=1}^{\infty} \frac{1}{e^{-n}+1}$ diverges by the Divergence Test.

31. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2+4}$. Let $f(x) = \frac{1}{(x+1)^2+4}$.
Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$I = \int_1^{\infty} \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \int_1^b \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{b+1}{2} - \frac{1}{2}\tan^{-1}\frac{1+1}{2}\right] = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$,
showing that the series converges.

33. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$. Let $f(x) = \frac{1}{x(\ln x)^p}$. If $p > 0$, then f is nonnegative, continuous, and decreasing on $[2, \infty)$.

$I = \int_2^{\infty} \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$. If $p > 1$, then the integral converges to $\frac{(\ln 2)^{1-p}}{p-1}$, and if $p < 1$, the integral diverges. If $p = 1$, $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln x} = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$.
Thus, I converges if $p > 1$, as does the series.

34. $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$. If $p < 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$, and so the series is divergent. For $p > 0$, let

$f(x) = \frac{\ln x}{x^p}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$. Using the table of integrals, we find

$\int_1^{\infty} \frac{\ln x}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^p}\, dx = \lim_{b\to\infty} \left\{\frac{b^{1-p}}{(1-p)^2}\left[-1 + (1-p)\ln b\right] + \frac{1}{(1-p)^2}\right\}$ for $p \neq 1$. If $p > 1$, then the integral converges, and if $p = 1$, then it diverges (see Exercise 21). If $0 \leq p < 1$, then the integral also diverges. Thus, the integral is convergent only for $p > 1$, as is the series.