3.6 Concept Questions

1. See page 307.

2. a. If $|x|$ is very large, then $\dfrac{1}{x^2}$ is very small, and so for large values of $|x|$, the graph of $f(x) = x^2 + \dfrac{1}{x^2}$ behaves like that of $g(x) = x^2$.

b. $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 + \dfrac{1}{x^2}\right) = \infty$ since the second term in the function grows arbitrarily large as x approaches 0.

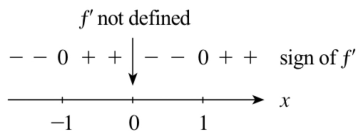
c. **(1)** The domain of f is $(-\infty, 0) \cup (0, \infty)$. **(2)** There is no x- or y-intercept. **(3)** $f(-x) = (-x)^2 + \dfrac{1}{(-x)^2} = x^2 + \dfrac{1}{x^2} = f(x)$, so the graph of f is symmetric with respect to the y-axis.

(4) $\lim_{x\to -\infty}\left(x^2 + \dfrac{1}{x^2}\right) = \lim_{x\to \infty}\left(x^2 + \dfrac{1}{x^2}\right) = \infty$

(5) $\lim_{x\to 0}\left(x^2 + \dfrac{1}{x^2}\right) = \infty$, and so $x = 0$ is a vertical asymptote.

There is no horizontal asymptote.

(6) $f'(x) = 2x - \dfrac{2}{x^3} = \dfrac{2\left(x^4 - 1\right)}{x^3} \Rightarrow \pm 1$ are critical numbers of f. From the sign diagram for f', we see that f is decreasing on $(-\infty, -1) \cup (0, 1)$ and increasing on $(-1, 0) \cup (1, \infty)$. **(7)** f has relative minima at $(-1, 2)$ and $(1, 2)$. **(8)** $f''(x) = 2 + \dfrac{6}{x^4} > 0$ for x in $(-\infty, 0) \cup (0, \infty)$, so f is concave upward on $(-\infty, 0) \cup (0, \infty)$. **(9)** f has no inflection point.



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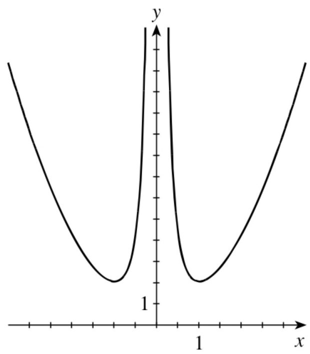
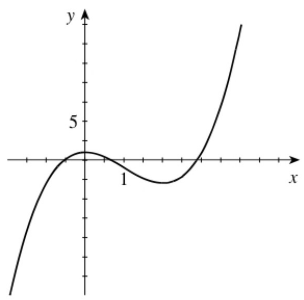


3.6 Curve Sketching

1.



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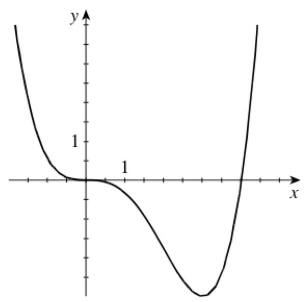

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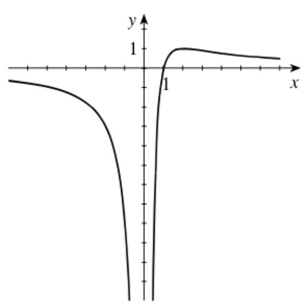

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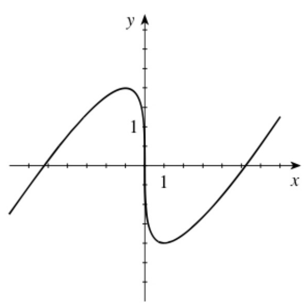
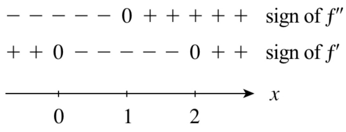


6. $f(x) = x^3 - 3x^2 + 2$



(1) The domain of f is $(-\infty, \infty)$. **(2)** The y-intercept is 2. The x-intercepts are not easily found, so we will not use this information. **(3)** There is no symmetry. **(4)** $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. **(5)** There is no asymptote. **(6)** $f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \Leftrightarrow x = 0$ or $x = 2$, so 0 and 2 are critical numbers of f. From the sign diagram for f', we see that f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$. **(7)** f has a relative maximum at $(0, 2)$ and a relative minimum at $(2, -2)$. **(8)** $f''(x) = 6x - 6 = 6(x-1) = 0 \Leftrightarrow x = 1$. From the sign diagram for f'', we see that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$. **(9)** f has an inflection point at $(1, 0)$.

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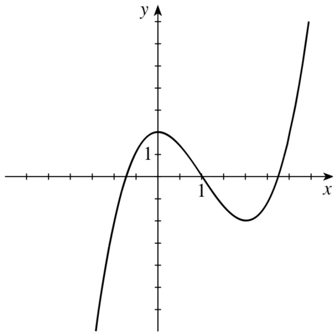
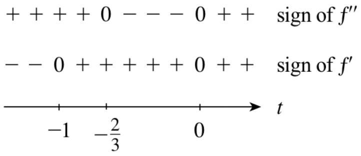


10. $f(t) = 3t^4 + 4t^3 = t^3(3t+4)$



(1) The domain of f is $(-\infty, \infty)$. **(2)** The y-intercept is 0. Setting $y = f(t) = 0$ gives 0 and $-\frac{4}{3}$ as the t-intercepts. **(3)** There is no symmetry. **(4)** $\lim_{t\to-\infty} f(t) = \infty$ and $\lim_{t\to\infty} f(t) = \infty$. **(5)** There is no asymptote. **(6)** $f'(t) = 12t^3 + 12t^2 = 12t^2(t+1)$. From the sign diagram for f', we see that f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$. **(7)** f has a relative minimum of $f(-1) = -1$. **(8)** $f''(t) = 36t^2 + 24t = 12t(3t+2) = 0 \Leftrightarrow t = -\frac{2}{3}$ or $t = 0$. From the sign diagram for f'', we see that f is concave upward on $\left(-\infty, -\frac{2}{3}\right)$ and $(0, \infty)$ and concave downward on $\left(-\frac{2}{3}, 0\right)$. **(9)** f has inflection points at $\left(-\frac{2}{3}, -\frac{16}{27}\right)$ and $(0, 0)$.

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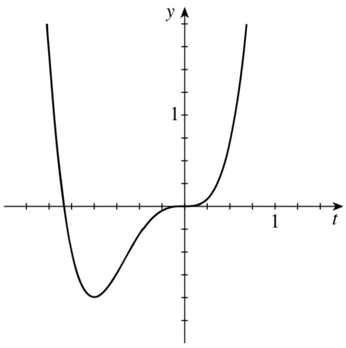
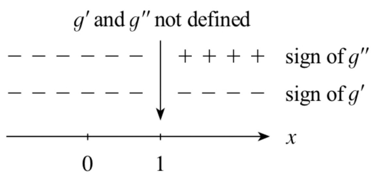


18. $g(x) = \dfrac{x+1}{x-1}$



(1) The domain of g is $(-\infty, 1) \cup (1, \infty)$. **(2)** The y-intercept is $g(0) = -1$ and the x-intercept is also -1. **(3)** There is no symmetry. **(4)** $\lim_{x\to-\infty} g(x) = \lim_{x\to\infty} g(x) = 1$. **(5)** $y = 1$ is a horizontal asymptote. Also, $\lim_{x\to1^-} g(x) = -\infty$ and $\lim_{x\to1^+} g(x) = \infty$, so $x = 1$ is a vertical asymptote. **(6)** $g'(x) = \dfrac{(x-1)-(x+1)}{(x-1)^2} = -\dfrac{2}{(x-1)^2}$. From the sign diagram for g, we see that g is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **(7)** g has no relative extremum because it has no critical number (1 is not in the domain of g). **(8)** $g''(x) = \dfrac{4}{(x-1)^3}$. From the sign diagram for g'', we see that g is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$. **(9)** There is no inflection point (1 is not in the domain of g).

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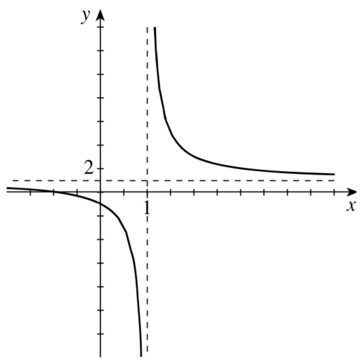
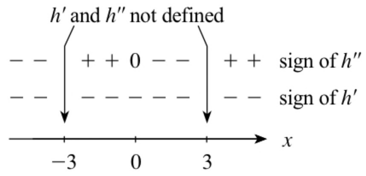


19. $h(x) = \dfrac{x}{x^2 - 9}$



(1) The domain of h is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. **(2)** The x- and y-intercepts are 0. **(3)** $h(-x) = \dfrac{-x}{(-x)^2 - 9} = -\dfrac{x}{x^2 - 9} = -h(x)$, so the graph of h is symmetric with respect to the origin.

(4) $\lim_{x \to -\infty} h(x) = 0$ and $\lim_{x \to \infty} h(x) = 0$. **(5)** From (4), we see that $y = 0$ is a horizontal asymptote. Since $\lim_{x \to -3^-} h(x) = \lim_{x \to 3^-} h(x) = -\infty$ and $\lim_{x \to -3^+} h(x) = \lim_{x \to 3^+} h(x) = \infty$, $x = \pm 3$ are vertical asymptotes.

(6) $h'(x) = \dfrac{\left(x^2 - 9\right) - x(2x)}{\left(x^2 - 9\right)^2} = -\dfrac{x^2 + 9}{\left(x^2 - 9\right)^2}$. From the sign diagram for h' we see that h is decreasing on its domain. **(7)** f has no relative extremum.

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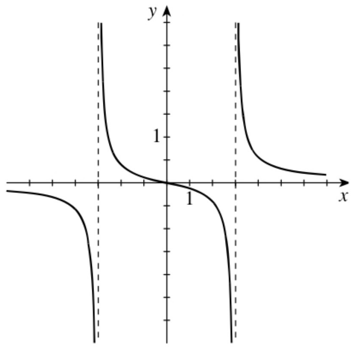
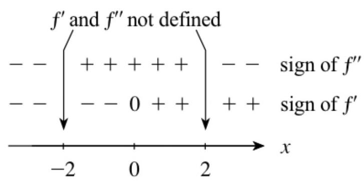


(8) $h''(x) = -\dfrac{\left(x^2 - 9\right)^2 (2x) - \left(x^2 + 9\right) 2\left(x^2 - 9\right)(2x)}{\left(x^2 - 9\right)^4} = \dfrac{2x\left(x^2 + 27\right)}{\left(x^2 - 9\right)^3}$. From the sign diagram of h'', we see that h is concave downward on $(-\infty, -3)$ and $(0, 3)$ and concave upward on $(-3, 0)$ and $(3, \infty)$. **(9)** h has an inflection point at $(0, 0)$. Neither of ± 3 is in the domain of h.

22. $g(x) = \dfrac{x^2 - 9}{x^2 - 4}$



(1) The domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. **(2)** The y-intercept is $\frac{9}{4}$ and the x-intercepts are ± 3. **(3)** Since $f(-x) = f(x)$, there is symmetry about the y-axis. **(4)** $\lim_{x \to \infty} \dfrac{x^2 - 9}{x^2 - 4} = \lim_{x \to \infty} \dfrac{1 - \frac{9}{x^2}}{1 - \frac{4}{x^2}} = 1$.

Similarly, $\lim_{x \to -\infty} \dfrac{x^2 - 9}{x^2 - 4} = 1$. **(5)** From the results of (4), we see that $y = 1$ is a horizontal asymptote. Now the denominator is 0 when $x^2 - 4 = 0 \Leftrightarrow x = \pm 2$. Since the numerator $x^2 - 9$ is not zero at $x = \pm 2$, we see that $x = \pm 2$ are vertical asymptotes.

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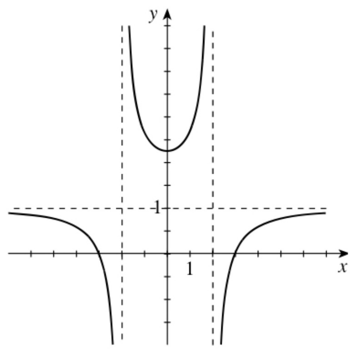


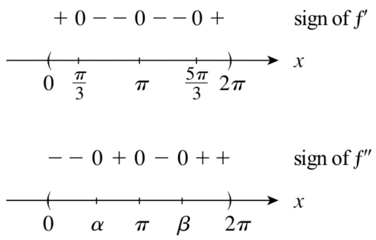
(6) $f'(x) = \dfrac{\left(x^2 - 4\right)(2x) - \left(x^2 - 9\right)(2x)}{\left(x^2 - 4\right)^2} = \dfrac{10x}{\left(x^2 - 4\right)^2} = 0 \Rightarrow x = 0$, and $f'(x)$ is discontinuous at $x = \pm 2$. From the sign diagram for f', we see that f is increasing on $(0, 2)$ and $(2, \infty)$ and decreasing on $(-\infty, -2)$ and $(-2, 0)$. **(7)** The point $\left(0, \frac{9}{4}\right)$ is a relative minimum.

(8) $f''(x) = \dfrac{\left(x^2 - 4\right)^2 (10) - (10x)(2)\left(x^2 - 4\right)(2x)}{\left(x^2 - 4\right)^4} = \dfrac{10\left(x^2 - 4\right)\left(x^2 - 4 - 4x^2\right)}{\left(x^2 - 4\right)^4} = \dfrac{-10\left(3x^2 + 4\right)}{\left(x^2 - 4\right)^3}$, which is not defined at $x = \pm 2$. From the sign diagram for f'', we see that f is concave upward on $(-2, 2)$ and concave downward on $(-\infty, -2)$ and $(2, \infty)$. **(9)** There is no inflection point. Note that $x = \pm 2$ are not in the domain of f.

26. $g(x) = 2\sin x + \sin 2x$, $0 \le x \le 2\pi$

(1) The domain of g is $[0, 2\pi]$. **(2)** The x-intercepts are 0, π, and 2π; the y-intercept is 0. **(3)** There is no symmetry. **(4)** Not applicable **(5)** There is no asymptote. **(6)** $f'(x) = 2\cos x + 2\cos 2x = 0 \Rightarrow x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$. From the sign diagram of f', we see that f is increasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$ and decreasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$.

(7) f has a relative maximum at $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}\right)$ and a relative minimum at $\left(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$. **(8)** $f''(x) = -2\sin x - 4\sin 2x = 0 \Rightarrow$ $x = \alpha = \cos^{-1}\left(-\frac{1}{4}\right) \approx 1.82$, π, or $\beta = 2\pi - \cos^{-1}\left(-\frac{1}{4}\right) \approx 4.46$. So f is concave downward on $(0, 1.82)$ and $(\pi, 4.46)$ and concave upward on $(1.82, \pi)$ and $(4.46, 2\pi)$. **(9)** From (8), we see that $(1.82, 1.45)$, $(\pi, 0)$, and $(4.46, -1.45)$ are inflection points.



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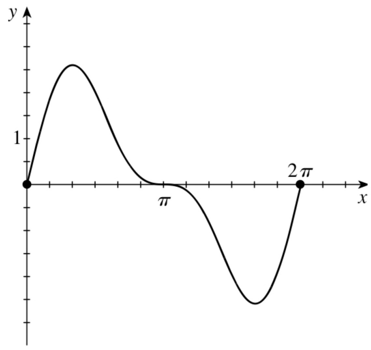

4.1 Concept Questions

1. An antiderivative of a function f on an interval I is a function F such that $F'(x) = f(x)$ for all x in I. For example, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $(-\infty, \infty)$.
2. $f(x) = g(x) + C$, where C is an arbitrary constant.
3. An antiderivative of f on I is a function F such that $F'(x) = f(x)$ for all x in I, whereas the indefinite integral of f is a family of antiderivatives of the form $F(x) + C$ on I, where $F' = f$ and C is an arbitrary constant.
4. See pages 355 and 356.

6. $\int \left(2x^{2/3} - 4x^{1/3} + 4\right) dx = \frac{6}{5}x^{5/3} - 3x^{4/3} + 4x + C$

14. $\int \frac{t^2 - 2\sqrt{t} + 1}{t^2}\, dt = \int \left(1 - 2t^{-3/2} + t^{-2}\right) dt = t + 4t^{-1/2} - t^{-1} + C = t + \frac{4}{\sqrt{t}} - \frac{1}{t} + C$

16. $\int \left(\pi^2 + \pi + 1\right) dx = \left(\pi^2 + \pi + 1\right) x + C$

22. $\int \sec u\,(\tan u + \sec u)\, du = \int \left(\sec u \tan u + \sec^2 u\right) du = \sec u + \tan u + C$

26. $\int \frac{\cos 2x}{\cos x - \sin x}\, dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}\, dx = \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{\cos x - \sin x}\, dx = \int (\cos x + \sin x)\, dx$

$= \sin x - \cos x + C$

28. $\int \tan^2 x\, dx = \int \left(\sec^2 x - 1\right) dx = \tan x - x + C$

29. $\int \frac{dx}{1 - \sin x} = \int \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 - \sin x}\, dx = \int \frac{1 + \sin x}{\cos^2 x}\, dx = \int \sec^2 x\, dx + \int \frac{\sin x}{\cos^2 x}\, dx$

$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx = \tan x + \sec x + C$

81. True, because $\frac{d}{dx}\left[f(x)+C\right]=f'(x)$, which is the integrand.

82. False. Let $F(x)=1$ and $G(x)=x$. Then $f(x)=F'(x)=0$ and $g(x)=G'(x)=1$, but $\int f(x)g(x)\,dx=\int 0\,dx=C$, where C is an arbitrary constant. But $F(x)G(x)+C=x+C\neq\int f(x)g(x)\,dx$.

83. False. Let $F(x)=x$, so that $f(x)=F'(x)=1$. Then $\int xf(x)\,dx=\int x(1)\,dx=\frac{1}{2}x^2+C_1\neq F(x)+C=x^2+C$.

84. True. By Rules 1 and 2 of Integration,

$$\begin{aligned}\int\left[2f(x)-3g(x)\right]dx &= \int 2f(x)\,dx-\int 3g(x)\,dx=2\int f(x)\,dx-3\int g(x)\,dx\\ &= 2F(x)+C_1-3G(x)+C_2 \text{ (where } F'=f \text{ and } G'=g)\\ &= 2F(x)-3G(x)+C\end{aligned}$$

85. False. Let $P(x)=2x$ and $Q(x)=4x^3$. Then $\int R(x)\,dx=\int\frac{2x}{4x^3}\,dx=\frac{1}{2}\int x^{-2}\,dx=-\frac{1}{2x}+C$, but

$$\frac{\int P(x)\,dx}{\int Q(x)\,dx}=\frac{\int 2x\,dx}{\int 4x^3\,dx}=\frac{x^2+C_1}{x^4+C_2}\neq-\frac{1}{2x}+C.$$

86. True, because $\frac{d}{dx}\left[G(x)+C_1x+C_2\right]=G'(x)+C_1=F(x)+C_1$ and $\frac{d}{dx}\left[F(x)+C_1\right]=f(x)$.