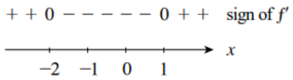
3.3 Increasing and Decreasing Functions and the First Derivative Test

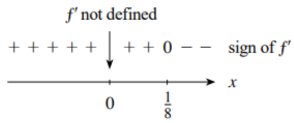
13. $f(x) = 2x^3 + 3x^2 - 12x + 5 \Rightarrow f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$ is continuous everywhere and has zeros at -2 and 1, the critical numbers of f. The sign diagram of f' is shown.



a. f is increasing on $(-\infty, -2)$ and $(1, \infty)$ and decreasing on $(-2, 1)$.

b. f has a relative maximum of $f(-2) = 25$ and a relative minimum value of $f(1) = -2$.

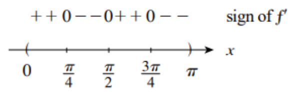
18. $f(x) = x^{1/3} - x^{2/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-1/3} = \frac{1}{3}x^{-2/3}\left(1 - 2x^{1/3}\right)$ is discontinuous at 0 and has a zero at $x = \frac{1}{8}$. The critical numbers of f are thus 0 and $\frac{1}{8}$. The sign diagram of f' is shown.



a. f is increasing on $\left(-\infty, \frac{1}{8}\right)$ and decreasing on $\left(\frac{1}{8}, \infty\right)$.

b. f has a relative maximum of $f\left(\frac{1}{8}\right) = \frac{1}{4}$.

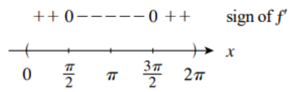
34. $f(x) = \sin^2 2x, 0 < x < \pi \Rightarrow$
$f'(x) = 2(\sin 2x \cos 2x)(2) = 4 \sin 2x \cos 2x = 2 \sin 4x$ is continuous and has zeros where $\sin 4x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ in $(0, \pi)$. The sign diagram of f' is shown.



a. f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and decreasing on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

b. f has relative maxima of $f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = 1$ and a relative minimum of $f\left(\frac{\pi}{2}\right) = 0$.

35. $f(x) = x \sin x + \cos x, 0 < x < 2\pi \Rightarrow f'(x) = \sin x + x \cos x - \sin x = x \cos x$ is continuous everywhere and has zeros where $x \cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ in $(0, 2\pi)$. The sign diagram of f' is shown.

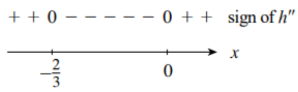


a. f is increasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ and decreasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

b. f has a relative maximum of $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ and a relative minimum of $f\left(\frac{3\pi}{2}\right) = -\frac{3\pi}{2}$.

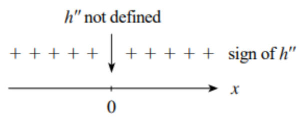
3.4 Concavity and Inflection Points

14. $h(x) = 3x^4 + 4x^3 + 1 \Rightarrow h'(x) = 12x^3 + 12x^2 \Rightarrow h''(x) = 36x^2 + 24x = 12x(3x+2)$.
The sign diagram of h'' is shown at right. We see that h is concave upward on $\left(-\infty, -\frac{2}{3}\right)$ and $(0, \infty)$ and concave downward on $\left(-\frac{2}{3}, 0\right)$. It has points of inflection at $\left(-\frac{2}{3}, \frac{11}{27}\right)$ and $(0, 1)$.



21. $h(x) = x^2 + x^{-2} \Rightarrow h'(x) = 2x - 2x^{-3} \Rightarrow h''(x) = 2 + 6x^{-4} = \dfrac{2\left(x^4 + 3\right)}{x^4}$.

The sign diagram of h'' is shown at right. We see that h is concave upward on $(-\infty, 0)$ and $(0, \infty)$. It has no inflection point.



44. $f(x) = x\sqrt{4-x^2} \Rightarrow f'(x) = \frac{1}{2}x\left(4-x^2\right)^{1/2}(-2x) + \left(4-x^2\right)^{1/2} = \dfrac{2\left(2-x^2\right)}{\left(4-x^2\right)^{1/2}} = 0 \Rightarrow x = \pm\sqrt{2}$,

the critical numbers of f. Note that f' is not defined at $x = \pm 2$, but these are the endpoints of the domain of f. $f''(x) = 2\left[\dfrac{\left(4-x^2\right)^{1/2}(-2x) - \left(2-x^2\right)\frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x)}{4-x^2}\right] = \dfrac{2x\left(x^2-6\right)}{\left(4-x^2\right)^{3/2}}$. We use the SDT:

$f''\left(-\sqrt{2}\right) = 4 > 0$, so f has a relative minimum of $f\left(-\sqrt{2}\right) = -2$; and $f''\left(\sqrt{2}\right) = -2 < 0$, so f has a relative maximum of $f\left(\sqrt{2}\right) = 2$.

45. $f(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2} \Rightarrow f'(x) = \cos x - \sin x = 0 \Leftrightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$ in $\left(0, \frac{\pi}{2}\right)$, so this is the only relevant critical number. $f''(x) = -\sin x - \cos x$, so using the SDT, we find that $f''\left(\frac{\pi}{4}\right) = -\sqrt{2} < 0$, implying that f has a relative maximum of $f\left(\frac{\pi}{4}\right) = \sqrt{2}$.