7.4 The Method of Partial Fractions

7. $I = \int \frac{dx}{x\,(x-4)}$. Now $\frac{1}{x\,(x-4)} = \frac{A}{x} + \frac{B}{x-4} = \frac{(A+B)\,x - 4A}{x\,(x-4)} \Rightarrow A + B = 0$ and $-4A = 1 \Rightarrow A = -\frac{1}{4}$ and $B = \frac{1}{4}$, so $I = -\frac{1}{4}\int \frac{dx}{x} + \frac{1}{4}\int \frac{dx}{x-4} = -\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4| + C = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$.

14. $I = \int_0^1 \frac{(2u+3)\,du}{u^2+4u+3}$. Now $\frac{2u+3}{u^2+4u+3} = \frac{2u+3}{(u+1)\,(u+3)} = \frac{A}{u+1} + \frac{B}{u+3} = \frac{(A+B)\,u + 3A + B}{(u+1)\,(u+3)}$ $\Rightarrow A + B = 2$ and $3A + B = 3 \Rightarrow A = \frac{1}{2}$ and $B = \frac{3}{2}$, so $I = \frac{1}{2}\int_0^1 \frac{du}{u+1} + \frac{3}{2}\int_0^1 \frac{du}{u+3} = \left(\frac{1}{2}\ln|u+1| + \frac{3}{2}\ln|u+3|\right)\Big|_0^1 = \frac{1}{2}\ln 2 + \frac{3}{2}\ln 4 - \frac{1}{2}\ln 1 - \frac{3}{2}\ln 3 = \frac{7}{2}\ln 2 - \frac{3}{2}\ln 3$ or $\frac{1}{2}\ln\frac{128}{27}$.

18. $I = \int_2^3 \frac{x^3 - 2x + 7}{x^2 + x - 2}\,dx$. The degree of the numerator is greater than that of the denominator; long division gives $I = \int_2^3 \left(x - 1 + \frac{x+5}{x^2+x-2}\right) dx$. Now $\frac{x+5}{x^2+x-2} = \frac{x+5}{(x+2)\,(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{(A+B)\,x - A + 2B}{(x+2)\,(x-1)} \Rightarrow$ $A + B = 1$ and $-A + 2B = 5$. Adding these equations gives $3B = 6 \Rightarrow B = 2$, and so $A = -1$. Thus,

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
x^2 + x - 2 & x^3 \qquad - 2x + 7 \\
 & x^3 + x^2 - 2x \\
\hline
 & -x^2 \qquad + 7 \\
 & -x^2 - x + 2 \\
\hline
 & x + 5
\end{array}
$$

$$I = \int_2^3 \left(x - 1 - \frac{1}{x+2} + \frac{2}{x-1}\right) dx = \left(\frac{1}{2}x^2 - x - \ln|x+2| + 2\ln|x-1|\right)\Big|_2^3$$
$$= \left(\frac{9}{2} - 3 - \ln 5 + 2\ln 2\right) - (2 - 2 - \ln 4) = \frac{3}{2} + \ln\frac{16}{5}$$

22. $I = \int_2^4 \frac{3x-5}{(x-1)^2}\,dx$. Now $\frac{3x-5}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A\,(x-1) + B}{(x-1)^2} = \frac{Ax + (B-A)}{(x-1)^2}$ $\Rightarrow A = 3$ and $B - A = -5$, so $B = -2$. Therefore,

$$I = \int_2^4 \left[\frac{3}{x-1} - \frac{2}{(x-1)^2}\right] dx = \left(3\ln|x-1| + \frac{2}{x-1}\right)\Big|_2^4 = \left(3\ln 3 + \frac{2}{3}\right) - (3\ln 1 + 2) = 3\ln 3 - \frac{4}{3}.$$

45. $I = \int \frac{\sin x\,dx}{\cos^3 x + \cos^2 x}$. Let $u = \cos x$, so $du = -\sin x\,dx$. Then $I = -\int \frac{du}{u^3 + u^2} = -\int \frac{du}{u^2\,(u+1)}$. Now $-\frac{1}{u^2\,(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} = \frac{(A+C)\,u^2 + (A+B)\,u + B}{u^2\,(u+1)}$, giving $A + C = 0$, $A + B = 0$, and $B = -1$. Solving, we find $A = 1$, $B = -1$, and $C = -1$. Thus,

$$I = \int \frac{du}{u} - \int \frac{du}{u^2} - \int \frac{du}{u+1} = \ln|u| + \frac{1}{u} - \ln|u+1| + C = \ln\left|\frac{u}{u+1}\right| + \frac{1}{u} + C = \ln\left|\frac{\cos x}{\cos x + 1}\right| + \sec x + C.$$

46. $I = \int \frac{\sec^2\theta\, d\theta}{\tan\theta\,(\tan\theta - 1)}$. Let $u = \tan\theta$, so $du = \sec^2\theta\, d\theta$. Then $I = \int \frac{du}{u\,(u-1)}$. Now $\frac{1}{u\,(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{(A+B)\,u - A}{u\,(u-1)}$, so $A + B = 0$ and $-A = 1 \Rightarrow B = 1$. Thus,

$I = -\int \frac{du}{u} + \int \frac{du}{u-1} = -\ln|u| + \ln|u-1| + C = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C.$

47. $I = \int \frac{e^t\, dt}{(e^t - 1)(e^t + 2)}$. Let $u = e^t$, so $du = e^t\, dt$. Then $I = \int \frac{du}{(u-1)(u+2)}$. Now $\frac{1}{(u-1)(u+2)} = \frac{A}{u-1} + \frac{B}{u+2} = \frac{(A+B)\,u + 2A - B}{(u-1)(u+1)}$, so $A + B = 0$ and $2A - B = 1$. Solving, we find $A = \frac{1}{3}$ and $B = -\frac{1}{3}$, so $I = \frac{1}{3}\int \frac{du}{u-1} - \frac{1}{3}\int \frac{du}{u+2} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+2| + C = \ln\left|\frac{u-1}{u+2}\right| + C = \frac{1}{3}\ln\left|\frac{e^t - 1}{e^t + 2}\right| + C.$

7.6 Improper Integrals

14. $\int_e^{\infty} \frac{dx}{x\ln^2 x} = \lim_{b\to\infty} \int_e^b \frac{(\ln x)^{-2}}{x}\, dx = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_e^b = \lim_{b\to\infty}\left(-\frac{1}{\ln b} + 1\right) = 1$

16. $I = \int_0^{\infty} e^{-x}\sin x\, dx = \lim_{b\to\infty} \int_0^b e^{-x}\sin x\, dx$. Using the result from Exercise 7.1.16, we find

$I = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-b}(\sin b + \cos b) + \frac{1}{2}\right] = \frac{1}{2}.$

17. $\int_0^{\infty} \frac{x\, dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{x\, dx}{1+x^2} = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(1+b^2\right)\right] = \infty$, so the integral diverges.

18.
$$\int_{-\infty}^0 \frac{dx}{x^2+2x+5} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{(x+1)^2+4} = \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+1)\right)\right]_a^0$$
$$= \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}(a+1)\right)\right] = \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{1}{2}\tan^{-1}\frac{1}{2} + \frac{\pi}{4}$$

21. $\int_{-\infty}^{\infty} \frac{e^x\, dx}{1+e^{2x}} = \lim_{a\to-\infty}\int_a^0 \frac{e^x\, dx}{1+e^{2x}} + \lim_{b\to\infty}\int_0^b \frac{e^x\, dx}{1+e^{2x}}$. Consider the indefinite integral $I = \int \frac{e^x\, dx}{1+e^{2x}} = \int \frac{e^x\, dx}{1+(e^x)^2}$. Let $u = e^x$, so $du = e^x\, dx$. Then $I = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1} e^x + C$. Using this result, we find $\int_{-\infty}^{\infty} \frac{e^x\, dx}{1+e^{2x}} = \lim_{a\to-\infty}\left[\tan^{-1} e^x\right]_a^0 + \lim_{b\to\infty}\left[\tan^{-1} e^x\right]_0^b = \lim_{a\to-\infty}\left(\frac{\pi}{4} - \tan^{-1} e^a\right) + \lim_{b\to\infty}\left(\tan^{-1} e^b - \frac{\pi}{4}\right) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

28. $\int_0^2 \frac{dx}{2x-3} = \int_0^{3/2} \frac{dx}{2x-3} + \int_{3/2}^2 \frac{dx}{2x-3}$. Since $\int_0^{3/2} \frac{dx}{2x-3} = \lim_{a\to(3/2)^-}\int_0^a \frac{dx}{2x-3} = \lim_{a\to(3/2)^-}\left[\frac{1}{2}\ln|2x-3|\right]_0^a = \lim_{a\to(3/2)^-}\left(\frac{1}{2}\ln|2a-3| - \frac{1}{2}\ln 3\right) = \infty$, we conclude that the integral is divergent.

30. $\int_0^2 \frac{dx}{x^2-2x} = \int_0^2 \frac{dx}{x\,(x-2)} = \int_0^2 \left(\frac{-1/2}{x} + \frac{1/2}{x-2}\right) dx = -\frac{1}{2}\int_0^2 \frac{dx}{x} + \frac{1}{2}\int_0^2 \frac{dx}{x-2}$. But $\int_0^2 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^2 \frac{dx}{x} = \lim_{a\to 0^+} [\ln x]_a^2 = \lim_{a\to 0^+} (\ln 2 - \ln a) = \infty$, so the integral is divergent.

32. Using the result of Example 6, we find

$$\begin{aligned}
\int_0^e \ln x\,dx &= \lim_{a\to 0^+} \int_a^e \ln x\,dx = \lim_{a\to 0^+} [x\ln x - x]_a^e = \lim_{a\to 0^+} [0 - (a\ln a - a)] \\
&= \lim_{a\to 0^+} [a\,(1-\ln a)] = \lim_{a\to 0^+} \frac{1-\ln a}{1/a} \qquad \text{(an indeterminate form; use l'Hôpital's Rule)} \\
&= \lim_{a\to 0^+} \frac{-1/a}{-1/a^2} = \lim_{a\to 0^+} a = 0
\end{aligned}$$