聯微作業解答 (6.1, 6.2)

6.1

- Differentiable the function.

34. $f(x) = \ln(x + \sqrt{x^2-1})$

Sol. $$f'(x) = \frac{1 + \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)}{x+\sqrt{x^2-1}} = \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$
$$= \frac{1}{\sqrt{x^2-1}}$$

39. $f(x) = \ln(x \ln x)$

Sol. $f'(x) = \dfrac{\ln x + 1}{x \ln x}$.

- Use logarithmic differentiation to find the derivative of the function.

67. $y = \sqrt[3]{\dfrac{x-1}{x^2+1}}$

Sol. $$y = \sqrt[3]{\frac{x-1}{x^2+1}} \Rightarrow \ln y = \frac{1}{3}\ln\frac{x-1}{x^2+1} = \frac{1}{3}(\ln(x-1) - \ln(x^2+1))$$
$$\Rightarrow \frac{y'}{y} = \frac{1}{3}(\frac{1}{x-1} - \frac{2x}{x^2+1}) = \frac{(x^2+1)-2x(x-1)}{3(x-1)(x^2+1)} = \frac{-x^2+2x+1}{3(x-1)(x^2+1)}$$
$$\Rightarrow y' = \frac{-x^2+2x+1}{3(x-1)^{\frac{2}{3}}(x^2+1)^{\frac{4}{3}}}$$

- Find or evaluate the integral.

82. $\int \dfrac{\sin 2x}{1+\sin^2 x} dx$

Sol. Let $u = 1+\sin^2 x$, $du = 2\sin x \cos x dx = \sin 2x dx$
$$\int \frac{\sin 2x}{1+\sin^2 x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln(1+\sin^2 x) + C,$$
C is a constant.

92. Let $y = \int_{\frac{2}{x}}^{x^2} \frac{dt}{t}$, where $x > 2$

a. Find $\frac{dy}{dx}$ by finding the integral and then differentiating the result.

b. Find $\frac{dy}{dx}$ using the Fundamental Theorem of Calculus, Part 1

Sol. a. $y = \int_{\frac{2}{x}}^{x^2} \frac{dt}{t} = \ln t\Big|_{\frac{2}{x}}^{x^2} = \ln x^2 - \ln \frac{2}{x} = 2\ln x - \ln 2 + \ln x$

$= 3\ln x - \ln 2$

$\frac{dy}{dx}(3\ln x - \ln 2) = \frac{3}{x}$

b. $\frac{dy}{dx} = \frac{d}{dx} \int_{\frac{2}{x}}^{x^2} \frac{dt}{t} = \frac{2x}{x^2} - \frac{\frac{-2}{x^2}}{\frac{2}{x}} = \frac{2}{x} + \frac{1}{x} = \frac{3}{x}$

6.2

• Find $f^{-1}(a)$ for the function f and the real number a.

22. $f(x) = 2x^5 + 3x^3 + 2$; $a = 2$

Sol. $f(0) = 2 \Rightarrow f^{-1}(2) = 0$

23. $f(x) = \dfrac{3}{\pi}x + \sin x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$; $a = 1$

Sol. $f(\frac{\pi}{6}) = \dfrac{1}{2} + \dfrac{1}{2} = 1 \Rightarrow f^{-1}(1) = \dfrac{\pi}{6}$

39. Let

$$f(x) = \begin{cases} 2x - 1, & \text{if} \quad x < 1 \\ \sqrt{x}, & \text{if} \quad 1 \le x < 4 \\ \frac{1}{2}x^2 - 6, & \text{if} \quad x \ge 4 \end{cases}$$

Find $f^{-1}(x)$, and state its domain.

Sol. Let $y = f^{-1}(x)$

If $x < 1$, $f(y) = 2y - 1 = x \Rightarrow y = \dfrac{x+1}{2}$, $y < 1$

$\Rightarrow f^{-1}(x) = \dfrac{x+1}{2}$, $x < 1$

If $1 \le x < 4$, $f(y) = \sqrt{y} = x \Rightarrow y = x^2$, $1 \le y < 2$

$\Rightarrow f^{-1}(x) = x^2$, $1 \le x < 2$

If $x \ge 4$, $f(y) = \frac{1}{2}y^2 - 6 = x \Rightarrow y = \sqrt{2x + 12}$, $y \ge 2$

$\Rightarrow f^{-1}(x) = \sqrt{2x + 12}$, $x \ge 2$

Hence

$$f^{-1}(x) = \begin{cases} \dfrac{x+1}{2}, & \text{if} \quad x < 1 \\ x^2, & \text{if} \quad 1 \le x < 2 \\ \sqrt{2x + 12}, & \text{if} \quad x \ge 2 \end{cases}$$

The domain of f^{-1} is $(-\infty, \infty)$

40. a. Show that $f(x) = -x^2 + x + 1$ on $[\frac{1}{2}, \infty)$ and $g(x) = \frac{1}{2} + \sqrt{\frac{5}{4} - x}$ on $(-\infty, \frac{5}{4})$ are inverse of each other.

b. Solve the equation $-x^2 + x + 1 = \frac{1}{2} + \sqrt{\frac{5}{4} - x}$

Hint: Use the result of part (a).

Sol. a. $f(g(x)) = f(\frac{1}{2} + \sqrt{\frac{5}{4} - x}) = -(\frac{1}{2} + \sqrt{\frac{5}{4} - x})^2 + (\frac{1}{2} + \sqrt{\frac{5}{4} - x}) + 1$

$= -(\frac{1}{4} + \sqrt{\frac{5}{4} - x} + \frac{5}{4} - x) + \frac{1}{2} + \sqrt{\frac{5}{4} - x} + 1 = x$

$g(f(x)) = g(-x^2 + x + 1) = \frac{1}{2} + \sqrt{\frac{5}{4} - (-x^2 + x + 1)}$

$= \frac{1}{2} + \sqrt{x^2 - x + \frac{1}{4}} = \frac{1}{2} + \sqrt{(x - \frac{1}{2})^2} = \frac{1}{2} + x - \frac{1}{2} = x$

b. $-x^2 + x + 1 = \frac{1}{2} + \sqrt{\frac{5}{4} - x} \Rightarrow f(x) = g(x)$. The equation is found by solving $y = f(x)$ and $y = g(x)$ at the same time. Since f and g are inverse of each other, f and g will intersect at $y = x$. Setting $f(x) = x \Rightarrow -x^2 + x + 1 = x \Rightarrow -x^2 + 1 = 0$ $\Rightarrow x = 1$ (Since -1 is not in the domain of f) Hence $x = 1$.

56. Suppose that g is the inverse of a differentiable function f and $H = g \circ g$. If $f(4) = 3$, $g(4) = 5$, $f'(4) = \frac{1}{2}$, and $f'(5) = 2$, find $H'(3)$.

Sol. $H'(x) = \dfrac{d}{dx}(g(g(x))) = g'(g(x))g'(x)$

$f(4) = 3 \Rightarrow g(3) = 4$, $g(4) = 5 \Rightarrow f(5) = 4$, $g' = \dfrac{1}{f'}$

$H'(3) = g'(g(3))g'(3) = g'(4)g'(3) = \dfrac{1}{f'(g(4))} \cdot \dfrac{1}{f'(g(3)))}$

$= \dfrac{1}{f'(5)} \cdot \dfrac{1}{f'(4)} = \frac{1}{2} \cdot \dfrac{1}{\frac{1}{2}} = 1$