9.4 The Comparison Tests

9. $\dfrac{\ln n}{n} > \dfrac{1}{n}$ for $n \geq 3$. Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$.

10. $\dfrac{\cos^2 n}{n^2} \leq \dfrac{1}{n^2}$ for $n \geq 1$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$.

11. $\dfrac{2 + \sin n}{3^n} < \dfrac{3}{3^n} = \dfrac{1}{3^{n-1}}$ for $n \geq 1$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{3^n}$ is convergent, so is $\displaystyle\sum_{n=1}^{\infty} \frac{2 + \sin n}{3^n}$.

14. If n is large, then $a_n = \dfrac{1}{\sqrt{n} + 2}$ behaves like $\dfrac{1}{\sqrt{n}} = b_n$. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+2} = 1 > 0$, so

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$ diverges.

16. If n is large, then $a_n = \dfrac{2n+1}{3n^2 - n + 1}$ behaves like $\dfrac{2n}{3n^2} = \dfrac{2}{3n}$, so take $b_n = \dfrac{1}{n}$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n+1}{3n^2-n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2+n}{3n^2-n+1} = \frac{2}{3} > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{3n^2-n+1}$ diverges.

20. If n is large, then $a_n = \dfrac{1}{2^n - 3}$ behaves like $b_n = \dfrac{1}{2^n}$. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-3}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-3} = 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

converges $\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n-3}$ converges.

22. If n is large, then $a_n = \dfrac{\ln n}{n^3 - 1}$ behaves like $b_n = \dfrac{\ln n}{n^3}$. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^3-1}}{\frac{\ln n}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1 > 0$. Since

$\dfrac{\ln n}{n^3} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, $\displaystyle\sum_{n=2}^{\infty} b_n$ converges by the Comparison Test, so the Limit Comparison Test implies that $\displaystyle\sum_{n=2}^{\infty} \frac{\ln}{n^3-1}$ converges.

26. $\dfrac{n}{\sqrt{n^5+n}} < \dfrac{n}{\sqrt{n^5}} = \dfrac{1}{n^{3/2}}$ for $n \geq 1$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n}}$ converges.

9.5 Alternating Series

2. $\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{3n-1}$ is an alternating series with $a_n = \frac{n}{3n-1}$, but $\lim_{n\to\infty} \frac{(-1)^n\, n}{3n-1}$ does not exist (the terms are close to $\pm\frac{1}{3}$ for large n, depending on the parity of n). Thus, the series diverges by the Divergence Test.

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n^2}{2n^2-1}$ is an alternating series, but $\lim_{n\to\infty} \frac{(-1)^{n-1}\, n^2}{2n^2-1}$ does not exist (the terms are close to $\pm\frac{1}{2}$ for large n, depending on the parity of n). Thus, the series diverges by the Divergence Test.

8. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln n}$ is an alternating series with $a_n = \frac{1}{\ln n}$. Since $a_{n+1} = \frac{1}{\ln(n+1)} < \frac{1}{\ln n} = a_n$ and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, the AST implies that the given series converges.

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{ne^{-n}}$ is an alternating series with $a_n = \frac{1}{ne^{-n}}$, but because $\lim_{n\to\infty} \frac{1}{ne^{-n}} = \lim_{n\to\infty} \frac{e^n}{n} = \infty$, $a_n = \frac{1}{ne^{-n}}$ does not approach 0, and thus the given series diverges by the Divergence Test .

26. $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$ is an alternating series with $a_n = \frac{(\ln n)^p}{n}$. If $p < 0$, then $\{a_n\}$ is eventually decreasing and $\lim_{n\to\infty} \frac{(\ln n)^p}{n} = 0$, so the series converges by the AST.

If $p = 0$, then $a_n = \frac{1}{n}$ and the series converges by the AST (see Example 1).

If $p > 0$, consider $f(x) = \frac{(\ln x)^p}{x} \Rightarrow f'(x) = \frac{xp(\ln x)^{p-1}(1/x) - (\ln x)^p}{x^2} = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ for $x > e^p$, so f (and therefore $\{a_n\}$) is eventually decreasing. To show that $\lim_{x\to\infty} \frac{(\ln x)^p}{x} = 0$, use l'Hôpital's Rule $p+1$ times (rounding down). For example, if $p = 2.2$, we have

$$\lim_{x\to\infty} \frac{(\ln x)^{2.2}}{x} = \lim_{x\to\infty} \frac{2.2(\ln x)^{1.2}\left(\frac{1}{x}\right)}{1} = \lim_{x\to\infty} \frac{2.2(\ln x)^{1.2}}{x} = \lim_{x\to\infty} \frac{2.2\cdot 1.2(\ln x)^{0.2}\left(\frac{1}{x}\right)}{1}$$

$$= \lim_{x\to\infty} \frac{2.2\cdot 1.2(\ln x)^{0.2}}{x} = \lim_{x\to\infty} \frac{2.2\cdot 1.2\cdot 0.2(\ln x)^{-0.8}\left(\frac{1}{x}\right)}{1} = 0$$

Thus, the given series converges for all real values of p by the AST.

4. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$. We use the Ratio Test on $a_n = \frac{(-1)^n\, 2^n}{n!}$, obtaining
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right] = \lim_{n\to\infty} \frac{2}{n+1} = 0$, so the series converges absolutely.

11. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$. Using the Ratio Test with $a_n = \frac{n!}{e^n}$, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right] = \lim_{n\to\infty} \frac{n+1}{e} = \infty$, so the series diverges.

15. $\sum_{n=1}^{\infty} \frac{2^n}{n!n}$. We use the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{2^{(n+1)}}{(n+1)!\,(n+1)} \cdot \frac{n!n}{2^n}\right] = \lim_{n\to\infty} \frac{2n}{(n+1)^2} = 0$, so the series converges.

19. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{2^n}$. Consider $\sum_{n=2}^{\infty} \left|\frac{(-1)^n \ln n}{2^n}\right| = \sum_{n=2}^{\infty} \frac{\ln n}{2^n}$. Since $\frac{\ln n}{2^n} < \frac{n}{2^n}$ and $\sum_{n=2}^{\infty} \frac{n}{2^n}$ converges (see Exercise 9.3.28), the Comparison Test implies that the given series converges absolutely.

24. $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^n$. We use the Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left[\left(\frac{\ln n}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$, so the series converges absolutely.