13.9 Lagrange Multipliers ET 12.9

8. $f(x,y)=8x+9y \Rightarrow \nabla f(x,y)=8\mathbf{i}+9\mathbf{j}$ and $g(x,y)=4x^2+9y^2-36 \Rightarrow \nabla g(x,y)=8x\mathbf{i}+18y\mathbf{j}$, so $\nabla f=\lambda\nabla g$ $\Rightarrow 8\mathbf{i}+9\mathbf{j}=8\lambda x\mathbf{i}+18\lambda y\mathbf{j}$, and we solve

$$\left.\begin{aligned} 8&=8\lambda x && (1)\\ 9&=18\lambda y && (2)\\ 4x^2+9y^2&=36 && (3)\end{aligned}\right\}$$

From (1), we see that $x=\dfrac{1}{\lambda}$, and from (2) we have $y=\dfrac{1}{2\lambda}$. Substituting into (3), we obtain $4\left(\dfrac{1}{\lambda}\right)^2+9\left(\dfrac{1}{2\lambda}\right)^2=36 \Leftrightarrow \lambda=\pm\frac{5}{12}$. If $\lambda=-\frac{5}{12}$, then $x=-\frac{12}{5}$ and $y=-\frac{6}{5}$, and if $\lambda=\frac{5}{12}$, then $x=\frac{12}{5}$ and $y=\frac{6}{5}$. We see that f has a minimum value of $f\left(-\frac{12}{5},-\frac{6}{5}\right)=-30$ and a maximum value of $f\left(\frac{12}{5},\frac{6}{5}\right)=30$.

11. $f(x,y,z)=x+2y+z \Rightarrow \nabla f(x,y,z)=\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $g(x,y,z)=x^2+4y^2-z \Rightarrow \nabla g(x,y,z)=2x\mathbf{i}+8y\mathbf{j}-\mathbf{k}$, so $\nabla f=\lambda\nabla g \Rightarrow \mathbf{i}+2\mathbf{j}+\mathbf{k}=2\lambda x\mathbf{i}+8\lambda y\mathbf{j}-\lambda\mathbf{k}$, and we solve

$$\left.\begin{aligned} 1&=2\lambda x\\ 2&=8\lambda y\\ 1&=-\lambda\\ x^2+4y^2-z&=0\end{aligned}\right\}$$

We find $\lambda=-1$, $x=-\frac{1}{2}$, $y=-\frac{1}{4}$, and $z=\frac{1}{2}$. We see that f has a minimum value of $f\left(-\frac{1}{2},-\frac{1}{4},\frac{1}{2}\right)=-\frac{1}{2}$.

17. $f(x,y,z)=2x+y \Rightarrow \nabla f(x,y,z)=2\mathbf{i}+\mathbf{j}$, $g(x,y,z)=x+y+z-1 \Rightarrow \nabla g(x,y,z)=\mathbf{i}+\mathbf{j}+\mathbf{k}$, and $h(x,y,z)=y^2+z^2-9 \Rightarrow \nabla h(x,y,z)=2y\mathbf{j}+2z\mathbf{k}$, so $\nabla f=\lambda\nabla g+\mu\nabla h$ along with the constraints $g(x,y,z)=0$ and $h(x,y,z)=0$ give the system

$$\left.\begin{aligned} 2&=\lambda && (1)\\ 1&=\lambda+2\mu y && (2)\\ 0&=\lambda+2\mu z && (3)\\ x+y+z&=1 && (4)\\ y^2+z^2&=9 && (5)\end{aligned}\right\}$$

From (1), (2), and (3), we obtain $\mu=-\dfrac{1}{2y}=-\dfrac{1}{z}$ $\Rightarrow z=2y$. Substituting this into (5) gives $y^2+(2y)^2=5y^2=9 \Leftrightarrow y=\pm\dfrac{3\sqrt{5}}{5} \Rightarrow z=\pm\dfrac{6\sqrt{5}}{5}$. Then (4) gives $x=1-\left(\pm\dfrac{3\sqrt{5}}{5}\right)-\left(\pm\dfrac{6\sqrt{5}}{5}\right)=\dfrac{5\mp 9\sqrt{5}}{5}$, so f has minimum value $f\left(\dfrac{5-9\sqrt{5}}{5},\dfrac{3\sqrt{5}}{5},\dfrac{6\sqrt{5}}{5}\right)=2-3\sqrt{5}$ and maximum value $f\left(\dfrac{5+9\sqrt{5}}{5},-\dfrac{3\sqrt{5}}{5},-\dfrac{6\sqrt{5}}{5}\right)=2+3\sqrt{5}$.

21. $$\left.\begin{aligned} f_x(x,y)&=\tfrac{\partial}{\partial x}\left(3x^2+2y^2-2x-1\right)=6x-2=0\\ f_y(x,y)&=\tfrac{\partial}{\partial y}\left(3x^2+2y^2-2x-1\right)=4y\quad=0\end{aligned}\right\}$$ $\Rightarrow x=\frac{1}{3}$ and $y=0$, so f has the critical point $\left(\frac{1}{3},0\right)$ in the disk $D=\left\{(x,y)\mid x^2+y^2\le 9\right\}$. Next, we use the method of Lagrange to find the critical points of f on the boundary of D. We write $g(x,y)=x^2+y^2-9=0$. Then $\nabla f(x,y)=(6x-2)\mathbf{i}+4y\mathbf{j}$ and $\nabla g(x,y)=2x\mathbf{i}+2y\mathbf{j}$. The equation $\nabla f=\lambda\nabla g$ and the constraint equation $g(x,y)=0$ give the system

$$\left.\begin{aligned} 6x-2&=2\lambda x && (1)\\ 4y&=2\lambda y && (2)\\ x^2+y^2&=9 && (3)\end{aligned}\right\}$$

Equation (2) gives $y=0$ or $\lambda=2$. If $y=0$, then (3) gives $x=\pm 3$; if $\lambda=2$, then (1) gives $x=1$. Substituting this value of x into (3) gives $y=\pm 2\sqrt{2}$.

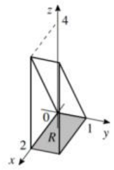
(x,y)	$\left(\frac{1}{3},0\right)$	$(-3,0)$	$(3,0)$	$\left(1,-2\sqrt{2}\right)$	$\left(1,2\sqrt{2}\right)$
$f(x,y)$	$-\frac{4}{3}$	32	20	16	16

From the table, we see that f has a minimum value of $f\left(\frac{1}{3},0\right)=-\frac{4}{3}$ and a maximum value of $f(-3,0)=32$.

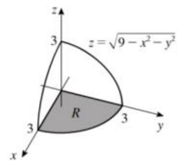
14 Multiple Integrals ET 13

14.1 Double Integrals ET 13.1

14. $\iint_R 2x\,dA$ represents the volume of the solid shown. Its base is $R = [0,2] \times [0,1]$, so
$\iint_R 2x\,dA = \frac{1}{2}\,(2)\,(1)\,(4) = 4.$



16. $\iint_R \,dA$ represents the volume of the solid shown. Its base is $R = \left\{(x,y) \mid x^2 + y^2 \le 9, x \ge 0, y \ge 0\right\}$ so, calculating one-eighth of the volume of a sphere with radius 3, $\iint_R \sqrt{9 - x^2 - y^2}\,dA = \frac{1}{8}\left[\frac{4}{3}\pi \cdot 3^3\right] = \frac{9\pi}{2}.$



25. Observe that $0 < e^{-1} \le e^{-x} \le 1$ for all x in $[0,1]$ and $0 < \cos y \le 1$ for all $0 \le y \le 1$, so $0 \le e^{-x}\cos y \le 1$ for all $(x,y) \in R$. Thus, by Property 4 of Theorem 1, we have $\iint_R 0\,dA \le \iint_R e^{-x}\cos y\,dA \le \iint_R 1\,dA \Leftrightarrow$
$0 \le \iint_R e^{-x}\cos y\,dA \le 1.$

26. Observe that $0 \le \sin(2x + 3y) \le 1$ on $R = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$, so by Property 4 of Theorem 1, we have
$\iint_R 0\,dA \le \iint_R \sin(2x + 3y)\,dA \le \iint_R 1\,dA = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$