9.2 Series

11. $\sum_{n=0}^{\infty} 2\left(-\frac{1}{\sqrt{2}}\right)^n = \frac{2}{1-\left(-\frac{1}{\sqrt{2}}\right)} = \frac{2}{1+\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = 2\sqrt{2}\left(\sqrt{2}-1\right) = 2\left(2-\sqrt{2}\right)$

17. Since $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$, the series diverges.

36. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2+n+1}{3n^2+2} = \lim_{n\to\infty} \frac{2+\frac{1}{n}+\frac{1}{n^2}}{3+\frac{2}{n^2}} = \frac{2}{3} \neq 0$, so $\sum_{n=0}^{\infty} \frac{2n^2+n+1}{3n^2+2}$ diverges.

38. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n-1}{3^{n+1}} = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{3^{n+1}}\right) = \frac{1}{3} \neq 0$, so $\sum_{n=1}^{\infty} \frac{3^n-1}{3^{n+1}}$ diverges.

9.3 The Integral Test

7. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{3/2}}$. Let $f(x) = \frac{x}{(x^2+1)^{3/2}}$. Then f is nonnegative, continuous, and decreasing on $[1,\infty)$.

$I = \int_1^{\infty} \frac{x\,dx}{(x^2+1)^{3/2}} = \lim_{b\to\infty} \int_1^b x\left(x^2+1\right)^{-3/2} dx = \lim_{b\to\infty} \left[-\left(x^2+1\right)^{-1/2}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$

Since I is convergent, so is the series.

27. $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \sum_{n=1}^{\infty} \left(\frac{-\frac{1}{2}}{2n+1} + \frac{\frac{1}{2}}{2n-1}\right)$ is a telescoping series:

$$S_n = \sum_{k=1}^{n} \left(\frac{-\frac{1}{2}}{2k+1} + \frac{\frac{1}{2}}{2k-1}\right) = \frac{1}{2}\left[\left(-\frac{1}{3}+1\right) + \left(-\frac{1}{5}+\frac{1}{3}\right) + \left(-\frac{1}{7}+\frac{1}{5}\right) + \cdots + \left(-\frac{1}{2n+1}+\frac{1}{2n-1}\right)\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \to \frac{1}{2} \text{ as } n\to\infty$$

and so $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ converges.

28. $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Let $f(x) = \frac{x}{2^x}$. Then f is nonnegative, continuous, and decreasing on $[2,\infty)$. To find $I = \int_1^{\infty} x2^{-x}\,dx$, let $u = x$ and $dv = 2^{-x}\,dx \Rightarrow du = dx$ and $v = -\frac{2^{-x}}{\ln 2}$. Then

$I = -\frac{x}{2^x \ln 2} + \frac{1}{\ln 2}\int 2^{-x}\,dx = -\frac{x}{2^x \ln 2} - \frac{1}{2^x (\ln 2)^2} + C = -\left(x + \frac{1}{\ln 2}\right)\frac{1}{2^x \ln 2} + C$, so

$\int_1^{\infty} x2^{-x}\,dx = \lim_{b\to\infty} I = \lim_{b\to\infty} \left[-\left(b + \frac{1}{\ln 2}\right)\left(\frac{1}{2^b \ln 2}\right) + \left(1 + \frac{1}{\ln 2}\right)\left(\frac{1}{2\ln 2}\right)\right] = \left(1 + \frac{1}{\ln 2}\right)\left(\frac{1}{2\ln 2}\right)$. Thus, the series converges.

31. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2+4}$. Let $f(x) = \frac{1}{(x+1)^2+4}$.

Then f is nonnegative, continuous, and decreasing on $[1,\infty)$.

$I = \int_1^{\infty} \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \int_1^b \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{b+1}{2} - \frac{1}{2}\tan^{-1}\frac{1+1}{2}\right] = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$,

showing that the series converges.