7.6 Improper Integrals

7. $\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty} \int_1^b x^{-3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}$

17. $\int_0^\infty \frac{x\,dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{x\,dx}{1+x^2} = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(1+b^2\right)\right] = \infty$, so the integral diverges.

23. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)^{3/2}} = \int_{-\infty}^0 \frac{x\,dx}{(x^2+1)^{3/2}} + \int_0^\infty \frac{x\,dx}{(x^2+1)^{3/2}} = \lim_{a\to-\infty}\int_a^0 \frac{x\,dx}{(x^2+1)^{3/2}} + \lim_{b\to\infty}\int_0^b \frac{x\,dx}{(x^2+1)^{3/2}}$

$$= \lim_{a\to-\infty}\left[\frac{1}{2}(-2)\left(x^2+1\right)^{-\frac{1}{2}}\right]_a^0 + \lim_{b\to\infty}\left[\frac{1}{2}(-2)\left(x^2+1\right)^{-\frac{1}{2}}\right]_0^b$$

$$= \lim_{a\to-\infty}\left(-1+\frac{1}{\sqrt{a^2+1}}\right) + \lim_{b\to\infty}\left(-\frac{1}{\sqrt{b^2+1}}+1\right) = -1+1=0$$

41. Working with the indefinite integral $I = \int \frac{\ln x}{\sqrt{x}}\,dx$, we use parts with $u=\ln x$ and $dv = x^{-1/2}\,dx \Rightarrow$ $du = dx/x$ and $v = 2\sqrt{x}$: $I = 2\sqrt{x}\ln x - \int 2x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C = 2\sqrt{x}(\ln x - 2) + C$, so $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[2\sqrt{x}(\ln x - 2)\right]_a^1 = \lim_{a\to 0^+}\left[-4 - 2\sqrt{a}(\ln a - 2)\right] = -4$ since, by l'Hôpital's Rule, $\lim_{a\to 0^+}\frac{\ln a}{a^{-1/2}} = \lim_{a\to 0^+}\frac{1/a}{-\frac{1}{2}a^{-3/2}} = \lim_{a\to 0^+}\left(-2\sqrt{a}\right) = 0$.

9 Infinite Series

9.1 Sequences

17. $\lim_{n\to\infty}\left(\frac{n-1}{n} - \frac{2n+1}{n^2}\right) = \lim_{n\to\infty}\frac{n^2-3n-1}{n^2} = \lim_{n\to\infty}\frac{1-\frac{3}{n}-\frac{1}{n^2}}{1} = 1$

19. $\lim_{n\to\infty}\frac{2n^2-3n+4}{3n^2+1} = \lim_{n\to\infty}\frac{2-\frac{3}{n}+\frac{4}{n^2}}{3+\frac{1}{n^2}} = \frac{2}{3}$

29. $\lim_{n\to\infty}\frac{\sin\sqrt{n}}{\sqrt{n}} = 0$ by the Squeeze Theorem: $-\frac{1}{\sqrt{n}} < \frac{\sin\sqrt{n}}{\sqrt{n}} < \frac{1}{\sqrt{n}}$ and $\lim_{n\to\infty}\left(\pm\frac{1}{\sqrt{n}}\right) = 0$, so the sequence converges to 0.

49. Put $f(x) = \frac{1-\left(1-\frac{1}{x}\right)^9}{1-\left(1-\frac{1}{x}\right)}$. Then

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1-\left(1-\frac{1}{x}\right)^9}{1-\left(1-\frac{1}{x}\right)} = \lim_{x\to\infty}\frac{-9\left(1-\frac{1}{x}\right)^8\left(\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$ (by l'Hôpital's Rule) $= \lim_{x\to\infty} 9\left(1-\frac{1}{x}\right)^8 = 9$, so by Theorem 1 we have $\lim_{x\to\infty}\frac{1-\left(1-\frac{1}{n}\right)^9}{1-\left(1-\frac{1}{n}\right)} = 9$.