聯合微積分第二次作業解答

1.4 Continuous Function

28. Let

$$f(x) = \begin{cases} \frac{x^2-4}{x+2}, & \text{if } x \neq -2 \\ k, & \text{if } x = -2 \end{cases}$$

Find the value of k that will make f continuous on $(-\infty, \infty)$.

假設 $f(x)$ 在 $x = -2$ 連續，則 $\lim_{x \to -2} f(x)$ 必須等於 k

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2-4}{x+2} = \lim_{x \to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x \to -2} (x-2) = -4$$

所以 $k = -4$

30. Let

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq 2 \\ kx^2-3, & \text{if } x > 2 \end{cases}$$

Find the value of k that will make f continuous on $(-\infty, \infty)$.

假設 $f(x)$ 在 $x = 2$ 連續，則 $\lim_{x \to -2} f(x)$ 必須存在且等於 $f(2)$

$$\begin{gathered} \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (kx+1) = 2k+1 \\ = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(kx^2-3\right) = 4k-3 \\ 4k-3 = 2k+1 \Rightarrow 2k = 4 \Rightarrow k = 2 \\ \lim_{x \to 2} f(x) = 5 \end{gathered}$$

當 $k = 2$ 時，

$$f(x) = \begin{cases} 2x+1, & \text{if } x \leq 2 \\ 2x^2-3, & \text{if } x > 2 \end{cases}$$

$$f(2) = 5 = \lim_{x \to 2} f(x)$$

因此當 $k = 2$ 時 $f(x)$ 連續.

60. Use the Intermediate Value Theorem to find the value of c such that $f(c) = M$.

$$f(x) = x^2 - 4x + 6 \ on\,[0, 3]\,; M = 3$$

將 $x = 0, 3$ 代入得到

$$f(0) = 0^2 - 4 \times 0 + 6 = 6$$
$$f(3) = 3^2 - 4 \times 3 + 6 = 3$$

由於 $f(3) = 3, f(0) = 6$

從 Intermediate Value Theorem (中間值定理)，我們無法知道在 $[0, 3]$ 是否有 $a \in (0, 3)$，使得 $f(a) = 3$

但是我們可以令 $x^2 - 4x + 6 = 3$

$$\begin{aligned} x^2 - 4x + 6 = 3 &\Rightarrow x^2 - 4x + 3 = 0 \\ &\Rightarrow (x - 3)(x - 1) = 0 \end{aligned}$$

透過解一元二次方程式找出 $f(1) = 1^2 - 4 \times 1 + 6 = 3$

64. Use Theorem 7 to show that there is at least one root of the equation in the given interval.

$$x^4 - 2x^3 - 3x^2 + 7 = 0; (1, 2)$$

設 $f(x) = x^4 - 2x^3 - 3x^2 + 7$, 則 $f(1) = 3, f(2) = -5$

由於 $f(1), f(2)$ 異號，所以 $f(x) = 0$ 在 $(1, 2)$ 有至少一個解.

90. Show that $f(x) = x^3 + x - 1$ has exactly one zero in (0,1).

$$f(0) = -1, f(1) = 1$$

$f(0), f(1)$ 異號，所以 $f(x) = 0$ 在 $(0, 1)$ 有至少一個解.

1.5 Tangent Lines and Rates of Change

16. Find the instantaneous rate of change of the given function when $x = a$

$$g(x) = x^2 - x + 2;\ a = -1$$

$$\lim_{h\to 0} \frac{g(-1+h) - g(-1)}{h} = \lim_{h\to 0} \frac{(-1+h)^2 - (-1+h) + 2 - 4}{h} = \lim_{h\to 0} \frac{[(-1+h) - 2][(-1+h) + 1]}{h}$$

$$= \lim_{h\to 0} \frac{h(h-3)}{h} = \lim_{h\to 0} (h-3) = -3$$

18. Find the instantaneous rate of change of the given function when $x = a$

$$f(x) = \sqrt{x},\ a = 4$$

$$\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h} = \lim_{h\to 0} \frac{(\sqrt{4+h}-\sqrt{4})(\sqrt{4+h}+\sqrt{4})}{h(\sqrt{4+h}+\sqrt{4})}$$
$$= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+\sqrt{4})} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+\sqrt{4})} = \lim_{h\to 0} \frac{1}{(\sqrt{4+h}+\sqrt{4})} = \frac{1}{4}$$

27.
a. Find the average rate of change of the area of a circle with respect to its radius r as r as increases from $r = 1$ to $r = 2$.

圓面積公式: πr^2

當 $r = 1$ 時圓面積為 π

當 $r = 2$ 時圓面積為 4π

$$\text{面積變化率:} \frac{4\pi - \pi}{2-1} = 3\pi$$

b. Find the rate of change of the area of a circle with respect to r when $r = 2$.

設 $f(r) = \pi r^2$

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2\pi - 2^2\pi}{h} = \lim_{h\to 0} \frac{[(2+h)+2][(2+h)-2]\pi}{h}$$
$$= \lim_{h\to 0} \frac{h(4+h)\pi}{h} = \lim_{h\to 0} (4+h)\pi = 4\pi$$

28.
a. Find the average of change of the volume of a sphere with respect to its radius r as r increases from $r = 1$ to $r = 2$.
球體積公式: $\frac{4}{3}\pi r^3$

當 $r = 1$ 時球體積為 $\frac{4}{3}\pi$

當 $r = 2$ 時球體積為 $\frac{16}{3}\pi$

$$\text{體積變化率:} \frac{\frac{16}{3}\pi - \frac{4}{3}\pi}{2-1} = 4\pi$$

b. Find the rate of change of the volume of a sphere with respect to r when $r = 2$.

設 $f(r) = \frac{4}{3}\pi r^3$

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(2+h)^3\pi - \frac{4}{3}\pi 2^3}{h} = \lim_{h\to 0} \frac{\frac{4}{3}\pi[(2+h)-2][(2+h)^2+2(2+h)+2^2]}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{3}\pi h(h^2+4h+4+2h+4+4)}{h} = \lim_{h\to 0} [\frac{4}{3}\pi(h^2+6h+12)] = \frac{48}{3}\pi = 16\pi$$