9.1

18. $\lim_{n\to\infty} \dfrac{n^2-1}{2n^2+1} = \lim_{n\to\infty} \dfrac{1-\frac{1}{n^2}}{2+\frac{1}{n^2}} = \dfrac{1}{2}$

22. $\lim_{n\to\infty} \dfrac{\sqrt{2n^2+1}}{n} = \lim_{n\to\infty} \dfrac{\sqrt{2+\frac{1}{n^2}}}{1} = \sqrt{2}$

24. $\lim_{n\to\infty} \left[1+\left(-\frac{2}{e}\right)^n\right] = 1+0 = 1$

32. $\lim_{n\to\infty} \dfrac{\ln n^2}{\sqrt{n}} = 0$. By l'Hôpital's Rule, $\lim_{x\to\infty} \dfrac{\ln x^2}{\sqrt{x}} = 2\lim_{x\to\infty} \dfrac{\ln x}{x^{1/2}} = 2\lim_{x\to\infty} \dfrac{1/x}{1/\left(2x^{1/2}\right)} = \lim_{x\to\infty} \dfrac{4}{x^{1/2}} = 0.$

33. $\lim_{n\to\infty} \dfrac{2^n}{3^n+1} = \lim_{n\to\infty} \dfrac{\left(\frac{2}{3}\right)^n}{1+\left(\frac{1}{3}\right)^n} = \dfrac{0}{1} = 0$

34. $\lim_{n\to\infty} \dfrac{2^n+1}{e^n} = \lim_{n\to\infty} \dfrac{\left(\frac{2}{e}\right)^n+\left(\frac{1}{e}\right)^n}{1} = \dfrac{0}{1} = 0$

36. $f(x) = \dfrac{x^p}{e^x}$ for $p > 0$. Using the result of Exercise 6.4.72, we see that $\lim_{x\to\infty} \dfrac{x^p}{e^x} = 0$, so $\lim_{n\to\infty} \dfrac{n^p}{e^n} = 0$.

37. $\lim_{x\to\infty} \left[\left(1+\dfrac{2}{x}\right)^{1/x}\right] = \lim_{u\to\infty} \left[\left(1+\dfrac{1}{u}\right)^{1/(2u)}\right]$ (where $u = \frac{1}{2}x$) $= 1^0 = 1$

9. 2

18. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$, so the series diverges.

20. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{n-1}} = \sum_{n=0}^{\infty} 2\left(-\frac{3}{2}\right)^n$ is a divergent geometric series since $|r| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$.

29. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left[\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}\right]$ is a telescoping series.

$S_n = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)\right] = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$, so

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{3}{4}$ and so $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$.

32. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{5^n} = 3\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{3}{1-\frac{3}{5}} = \frac{15}{2}$

39. $\sum_{n=1}^{\infty} \left[\frac{1}{2^n} - \frac{1}{n(n+1)}\right] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Now $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ by Example 9.2.1b, so $\sum_{n=1}^{\infty} \left[\frac{1}{2^n} - \frac{1}{n(n+1)}\right] = 1 - 1 = 0$.

42. $S_n = \sum_{k=1}^{n} \ln\frac{k}{k+1} = \sum_{k=1}^{n} [\ln k - \ln(k+1)] = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + [\ln n - \ln(n+1)] = -\ln(n+1)$.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [-\ln(n+1)] = -\infty$, so $\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$ diverges.

47. $\sum_{n=0}^{\infty} \frac{2^n+3^n}{6^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{1}{2}} = \frac{3}{2} + 2 = \frac{7}{2}$

49. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \neq 0$, so by the Divergence Test, $\sum_{n=1}^{\infty} \tan^{-1} n$ diverges.

54. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = e^2 \neq 0$, so $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^n$ diverges by the Divergence Test.

9. 3

16. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^2+1}}$. Let $f(x) = \frac{x}{\sqrt{2x^2+1}}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$I = \int_1^{\infty} \frac{x\,dx}{\sqrt{2x^2+1}} = \lim_{b\to\infty} \int_1^b x\left(2x^2+1\right)^{-1/2} dx = \lim_{b\to\infty}\left(\frac{1}{2}\sqrt{2b^2+1} - \frac{1}{2}\sqrt{3}\right) = \infty$. Since I diverges, so does the series.

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21. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$. Let $f(x) = \frac{\ln x}{x}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$. But

$\int_2^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_2^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 2)^2\right] = \infty$, so the series diverges.

22. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$. Let $f(x) = \frac{\ln x}{x^2}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$. We calculate $I = \int_2^{\infty} \frac{\ln x}{x^2}\,dx$ using the substitution $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $x = e^u$. Then $I = \int \frac{u\,du}{e^u} = \int ue^{-u}\,du = -(1+u)e^{-u} + C$ (by parts).

Therefore, $I = \lim_{b\to\infty} \int_2^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1+\ln x}{x}\right]_2^b = \lim_{b\to\infty}\left[-\frac{1+\ln b}{b} + \frac{1+\ln 2}{2}\right] = \frac{1+\ln 2}{2}$ by l'Hôpital's Rule. Thus, the series converges.

23. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Let $f(x) = \frac{1}{x(\ln x)^2}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\left[-\frac{1}{\ln b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$, so the series converges.

24. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$. Let $f(x) = \frac{e^{1/x}}{x^2}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{e^{1/x}}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{e^{1/x}}{x^2}\,dx = \lim_{b\to\infty}\left(-e^{1/b} + e\right) = e - 1$, so the series converges.

30. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{e^{-n}+1} = 1 \neq 0$, so $\sum_{n=1}^{\infty} \frac{1}{e^{-n}+1}$ diverges by the Divergence Test.

31. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2+4}$. Let $f(x) = \frac{1}{(x+1)^2+4}$.

Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$I = \int_1^{\infty} \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \int_1^b \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{b+1}{2} - \frac{1}{2}\tan^{-1}\frac{1+1}{2}\right] = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$, showing that the series converges.

33. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$. Let $f(x) = \dfrac{1}{x(\ln x)^p}$. If $p > 0$, then f is nonnegative, continuous, and decreasing on $[2, \infty)$.

$I = \displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right]$. If $p > 1$, then the integral converges to $\dfrac{(\ln 2)^{1-p}}{p-1}$, and if $p < 1$, the integral diverges. If $p = 1$, $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln x} = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$. Thus, I converges if $p > 1$, as does the series.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$. If $p < 0$, then $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$, and so the series is divergent. For $p > 0$, let $f(x) = \dfrac{\ln x}{x^p}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$. Using the table of integrals, we find $\displaystyle\int_1^{\infty} \frac{\ln x}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^p}\,dx = \lim_{b\to\infty} \left\{ \frac{b^{1-p}}{(1-p)^2} \left[-1 + (1-p)\ln b\right] + \frac{1}{(1-p)^2} \right\}$ for $p \neq 1$. If $p > 1$, then the integral converges, and if $p = 1$, then it diverges (see Exercise 21). If $0 \leq p < 1$, then the integral also diverges. Thus, the integral is convergent only for $p > 1$, as is the series.