13. 8

9. $$\left.\begin{aligned} f_x(x,y) &= \tfrac{\partial}{\partial x}\left(x^2+2y^2+x^2y+3\right) = 2x+2xy = 0 \\ f_y(x,y) &= \tfrac{\partial}{\partial y}\left(x^2+2y^2+x^2y+3\right) = 4y+x^2 = 0 \end{aligned}\right\}$$ The first equation gives $x = 0$ or $y = -1$. Substituting $x = 0$ into the second equation gives $y = 0$; substituting $y = -1$ into the second equation gives $x = \pm 2$. Thus, f has critical points $(0, 0)$, $(-2, -1)$, and $(2, -1)$. Next, $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - f_{xy}^2(x, y) = (2 + 2y)(4) - (2x)^2 = -4x^2 + 8y + 8$.

At $(0, 0)$: $D(0, 0) = 8 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so $(0, 0)$ gives a relative minimum of f with value $f(0, 0) = 3$.

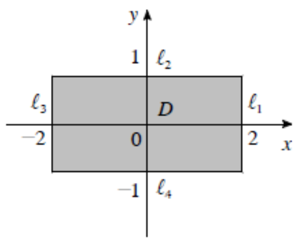
At $(-2, -1)$: $D(-2, -1) = -4(-2)^2 + 8(-1) + 8 = -16 < 0$, so $(-2, -1, 5)$ is a saddle point of f.

At $(2, -1)$: $D(2, -1) = -4(2)^2 + 8(-1) + 8 = -16 < 0$, so $(2, -1, 5)$ is also a saddle point of f.

13. $$\left.\begin{aligned} f_x(x,y) &= \tfrac{\partial}{\partial x}\left(x^2-6x-x\sqrt{y}+y\right) = 2x-6-\sqrt{y} = 0 \\ f_y(x,y) &= \tfrac{\partial}{\partial y}\left(x^2-6x-xy^{1/2}+y\right) = -\frac{x}{2\sqrt{y}}+1 = 0 \end{aligned}\right\}$$ From the first equation, we see that $\sqrt{y} = 2x - 6$. Substituting this into the second equation gives $-x + 2(2x - 6) = 0 \Rightarrow x = 4$. Substituting this into the first equation gives $8 - 6 = \sqrt{y} \Rightarrow y = 4$, so the sole critical point of f is $(4, 4)$. Next,

$D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - f_{xy}^2(x, y) = 2\left(\frac{x}{4y^{3/2}}\right) - \left(-\frac{1}{2\sqrt{y}}\right)^2 = \frac{x}{2y^{3/2}} - \frac{1}{4y}$. Since $D(4, 4) = \frac{4}{2(8)} - \frac{1}{4(4)} = \frac{3}{16} > 0$ and $f_{xx}(4, 4) = 2 > 0$, the point $(4, 4)$ gives a relative minimum of f with value $f(4, 4) = -12$.

34. $$\left.\begin{aligned} f_x(x,y) &= \tfrac{\partial}{\partial x}\left(x^2+xy+y^2\right) = 2x+y = 0 \\ f_y(x,y) &= \tfrac{\partial}{\partial y}\left(x^2+xy+y^2\right) = x+2y = 0 \end{aligned}\right\} \Rightarrow x = 0 \text{ and } y = 0,$$



so $(0, 0)$ is the only critical point of f in D.

On ℓ_1, $x = 2$ and $y = y$, so $g(y) = f(2, y) = 4 + 2y + y^2$ for $-1 \le y \le 1$. $g'(y) = 2 + 2y = 0 \Rightarrow y = -1$, so since $g(-1) = 3$ and $g(1) = 7$, we see that f has an absolute minimum value of 3 and an absolute maximum value of 7 on ℓ_1.

On ℓ_2, $x = x$ and $y = 1$, so $h(x) = f(x, 1) = x^2 + x + 1$ for $-2 \le x \le 2$. $h'(x) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$, so $-\frac{1}{2}$ is a critical number of h. Since $h(-2) = 3$, $h\left(-\frac{1}{2}\right) = \frac{3}{4}$, and $h(2) = 7$, we see that f has an absolute minimum value of $\frac{3}{4}$ and an absolute maximum value of 7 on ℓ_2.

On ℓ_3, $x = -2$ and $y = y$, so $s(y) = f(-2, y) = 4 - 2y + y^2$ for $-1 \le y \le 1$. $s'(y) = -2 + 2y = 0 \Rightarrow y = 1$, an endpoint. Since $s(-1) = 7$ and $s(1) = 3$, we see that f has an absolute minimum value of 3 and an absolute maximum value of 7 on ℓ_3.

On ℓ_4, $x = x$ and $y = -1$, so $t(x) = f(x, -1) = x^2 - x + 1$ for $-2 \le x \le 2$. $g'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$, so t has critical number $\frac{1}{2}$. Since $g(-2) = 7$, $g\left(\frac{1}{2}\right) = \frac{3}{4}$, and $g(2) = 3$, we see that f has an absolute minimum value of $\frac{3}{4}$ and an absolute maximum value of 7 on ℓ_4.

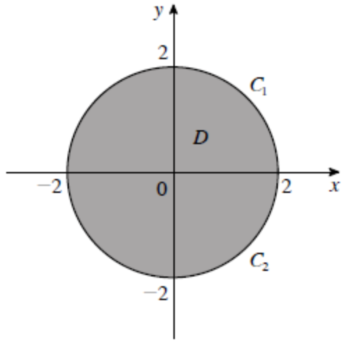
The extreme values of f on D and its boundaries are summarized below.

	Critical point	ℓ_1		ℓ_2		ℓ_3		ℓ_4	
(x, y)	$(0, 0)$	$(2, -1)$	$(2, 1)$	$\left(-\frac{1}{2}, 1\right)$	$(2, 1)$	$(-2, -1)$	$(-2, 1)$	$\left(\frac{1}{2}, -1\right)$	$(-2, -1)$
Extreme value	0	3	7	$\frac{3}{4}$	7	7	3	$\frac{3}{4}$	7

We see that f has an absolute minimum value of 0 and an absolute maximum value of 7 on D.

39. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2+4y^2+3x-1\right) = 2x+3 = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2+4y^2+3x-1\right) = 8y = 0 \end{array}\right\} \Rightarrow x = -\tfrac{3}{2} \text{ and}$$

$y = 0$, so f has the critical point $\left(-\frac{3}{2}, 0\right)$ in D with $f\left(-\frac{3}{2}, 0\right) = -\frac{13}{4}$.



Next we consider the boundary of D. On C_1, $y = \sqrt{4-x^2}$, so

$g(x) = f\left(x, \sqrt{4-x^2}\right) = x^2 + 4\left(\sqrt{4-x^2}\right)^2 + 3x - 1 = -3x^2 + 3x + 15$

for $-2 \le x \le 2$. $g'(x) = -6x + 3 = 0 \Rightarrow x = \frac{1}{2}$, so f has a critical point at $\left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right)$. From the table, we see that f has an absolute minimum value of -3 and an absolute maximum value of $\frac{63}{4}$ on C_1.

x	-2	$\frac{1}{2}$	2
$g(x)$	-3	$\frac{63}{4}$	9

On C_2, $y = -\sqrt{4-x^2}$, so $h(x) = f\left(x, -\sqrt{4-x^2}\right) = x^2 + 4\left(-\sqrt{4-x^2}\right)^2 + 3x - 1$, which is the same as $g(x)$. So h has a critical point at $\left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$. We conclude that f has an absolute minimum value of $-\frac{13}{4}$ attained at the critical point $\left(-\frac{3}{2}, 0\right)$ on D, and an absolute maximum value of $\frac{63}{4}$ attained at the points $\left(\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right)$ on the boundary of D.

57. $g(m,b) = \sum_{k=1}^{n}(y_k - mx_k - b)^2$, so we require that $$\left.\begin{array}{l} g_m(m,b) = -2\sum_{k=1}^{n} x_k(y_k - mx_k - b) = 0 \\ g_b(m,b) = -2\sum_{k=1}^{n}(y_k - mx_k - b) \quad = 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \left(\sum_{k=1}^{n} x_k^2\right)m + \left(\sum_{k=1}^{n} x_k\right)b = \sum_{k=1}^{n} x_k y_k \\ \left(\sum_{k=1}^{n} x_k\right)m + \left(\sum_{k=1}^{n} 1\right)b = \sum_{k=1}^{n} y_k \end{array}\right\} \Rightarrow \left.\begin{array}{l} \left(\sum_{k=1}^{n} x_k^2\right)m + \left(\sum_{k=1}^{n} x_k\right)b = \sum_{k=1}^{n} x_k y_k \\ \left(\sum_{k=1}^{n} x_k\right)m + nb = \sum_{k=1}^{n} y_k \end{array}\right\}$$ It is clear that a minimum value of g must exist, so the critical point (m_0, b_0) satisfying the system gives the least-squares line $y = m_0 x + b_0$ that we seek.

13.9

8. $f(x,y) = 8x + 9y \Rightarrow \nabla f(x,y) = 8\mathbf{i} + 9\mathbf{j}$ and $g(x,y) = 4x^2 + 9y^2 - 36 \Rightarrow \nabla g(x,y) = 8x\mathbf{i} + 18y\mathbf{j}$, so $\nabla f = \lambda \nabla g$

$\Rightarrow 8\mathbf{i} + 9\mathbf{j} = 8\lambda x\mathbf{i} + 18\lambda y\mathbf{j}$, and we solve $\left.\begin{aligned} 8 &= 8\lambda x & (1) \\ 9 &= 18\lambda y & (2) \\ 4x^2 + 9y^2 &= 36 & (3) \end{aligned}\right\}$ From (1), we see that $x = \dfrac{1}{\lambda}$, and from (2) we have $y = \dfrac{1}{2\lambda}$. Substituting into (3), we obtain $4\left(\dfrac{1}{\lambda}\right)^2 + 9\left(\dfrac{1}{2\lambda}\right)^2 = 36 \Leftrightarrow \lambda = \pm\frac{5}{12}$. If $\lambda = -\frac{5}{12}$, then $x = -\frac{12}{5}$ and $y = -\frac{6}{5}$, and if $\lambda = \frac{5}{12}$, then $x = \frac{12}{5}$ and $y = \frac{6}{5}$. We see that f has a minimum value of $f\left(-\frac{12}{5}, -\frac{6}{5}\right) = -30$ and a maximum value of $f\left(\frac{12}{5}, \frac{6}{5}\right) = 30$.

10. $f(x,y) = x^2 + y^2 \Rightarrow \nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$ and $g(x,y) = x^4 + y^4 - 1 \Rightarrow \nabla g(x,y) = 4x^3\mathbf{i} + 4y^3\mathbf{j}$, so $\nabla f = \lambda \nabla g$

$\Rightarrow 2x\mathbf{i} + 2y\mathbf{j} = 4\lambda x^3\mathbf{i} + 4\lambda y^3\mathbf{j}$, and we solve $\left.\begin{aligned} 2x &= 4\lambda x^3 \\ 2y &= 4\lambda y^3 \\ x^4 + y^4 &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x\left(2\lambda x^2 - 1\right) &= 0 & (1) \\ y\left(2\lambda y^2 - 1\right) &= 0 & (2) \\ x^4 + y^4 &= 1 & (3) \end{aligned}\right\}$ From (1), we see that $x = 0$ or $x^2 = \dfrac{1}{2\lambda}$. From (2), we have $y = 0$ or $y^2 = \dfrac{1}{2\lambda}$. If $x = 0$, then (3) gives $y = \pm 1$ and if $y = 0$, then $x = \pm 1$. If $x^2 = \dfrac{1}{2\lambda}$ and $y^2 = \dfrac{1}{2\lambda}$, then (3) gives $\left(\dfrac{1}{2\lambda}\right)^2 + \left(\dfrac{1}{2\lambda}\right)^2 = 1 \Rightarrow \lambda = \pm\dfrac{1}{\sqrt{2}}$. Now if $\lambda = \dfrac{1}{\sqrt{2}}$, then $x = \pm\dfrac{1}{\sqrt[4]{2}}$ and $y = \pm\dfrac{1}{\sqrt[4]{2}}$.

(x,y)	$(0,-1)$	$(0,1)$	$(-1,0)$	$(1,0)$	$\left(-\sqrt[4]{2}, -\sqrt[4]{2}\right)$	$\left(-\sqrt[4]{2}, \sqrt[4]{2}\right)$	$\left(\sqrt[4]{2}, -\sqrt[4]{2}\right)$	$\left(\sqrt[4]{2}, \sqrt[4]{2}\right)$
$f(x,y)$	1	1	1	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$

From the table, we see that f has a minimum value of 1 and a maximum value of $\sqrt{2}$.

19. $f(x,y,z) = yz + xz \Rightarrow \nabla f(x,y,z) = z\mathbf{i} + z\mathbf{j} + (x+y)\mathbf{k}$, $g(x,y,z) = xz - 1 \Rightarrow \nabla g(x,y,z) = z\mathbf{i} + x\mathbf{k}$, and $h(x,y,z) = y^2 + z^2 - 1 \Rightarrow \nabla h(x,y,z) = 2y\mathbf{j} + 2z\mathbf{k}$, so $\nabla f = \lambda\nabla g + \mu\nabla h$ along with the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$ give the system $\left.\begin{aligned} z &= \lambda z & (1) \\ z &= 2\mu y & (2) \\ x + y &= \lambda x + 2\mu z & (3) \\ xz &= 1 & (4) \\ y^2 + z^2 &= 1 & (5) \end{aligned}\right\}$ From (1) we have $\lambda = 1$ since $z \neq 0$ by (4). From (2) we have $\mu = \dfrac{z}{2y}$. Substituting these into (3) gives $x + y = x + 2\left(\dfrac{z}{2y}\right)z \Rightarrow y^2 = z^2 \Rightarrow z = \pm y$. Using (5), we obtain $y^2 + y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{2}}{2} \Rightarrow z = \pm\frac{\sqrt{2}}{2}$, and by (4), $x = \pm\sqrt{2}$. Because x and z must have the same sign by (4), we consider the points $\left(\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\sqrt{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(-\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. We see that f has minimum value $f\left(-\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = f\left(\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2}$ and maximum value $f\left(-\sqrt{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = f\left(\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{3}{2}$.

20. $f(x,y,z) = x^2+y^2+z^2 \Rightarrow \nabla f(x,y,z) = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$, $g(x,y,z) = 2x+y+z-2 \Rightarrow \nabla g(x,y,z) = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$, and $h(x,y,z) = x-2y+3z+4 = 0 \Rightarrow \nabla h(x,y,z) = \mathbf{i}-2\mathbf{j}+3\mathbf{k}$, so $\nabla f = \lambda\nabla g + \mu\nabla h$ along with the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$ give the system

$$\left.\begin{array}{rl} 2x = 2\lambda+\mu & (1) \\ 2y = \lambda-2\mu & (2) \\ 2z = \lambda+3\mu & (3) \\ 2x+y+z = 2 & (4) \\ -x+2y-3z = 4 & (5) \end{array}\right\}$$

Solving the system, we find $\lambda = \frac{16}{15}$, $\mu = -\frac{4}{5}$, $x = \frac{2}{3}$, $y = \frac{4}{3}$, and $z = -\frac{2}{3}$. We see that the minimum value of f is $f\left(\frac{2}{3},\frac{4}{3},-\frac{2}{3}\right) = \frac{8}{3}$. Note that f is unbounded above since the two constraints imply that f is restricted to lie on an unbounded line.

21. $$\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(3x^2+2y^2-2x-1\right) = 6x-2 = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(3x^2+2y^2-2x-1\right) = 4y \quad = 0 \end{array}\right\} \Rightarrow x = \frac{1}{3} \text{ and } y = 0,$$ so f has the critical point $\left(\frac{1}{3},0\right)$ in the disk $D = \left\{(x,y) \mid x^2+y^2 \le 9\right\}$. Next, we use the method of Lagrange to find the critical points of f on the boundary of D. We write $g(x,y) = x^2+y^2-9 = 0$. Then $\nabla f(x,y) = (6x-2)\mathbf{i}+4y\mathbf{j}$ and $\nabla g(x,y) = 2x\mathbf{i}+2y\mathbf{j}$. The equation $\nabla f = \lambda\nabla g$ and the constraint equation $g(x,y) = 0$ give the system

$$\left.\begin{array}{rl} 6x-2 = 2\lambda x & (1) \\ 4y = 2\lambda y & (2) \\ x^2+y^2 = 9 & (3) \end{array}\right\}$$

Equation (2) gives $y = 0$ or $\lambda = 2$. If $y = 0$, then (3) gives $x = \pm 3$; if $\lambda = 2$, then (1) gives $x = 1$. Substituting this value of x into (3) gives $y = \pm 2\sqrt{2}$.

(x,y)	$\left(\frac{1}{3},0\right)$	$(-3,0)$	$(3,0)$	$\left(1,-2\sqrt{2}\right)$	$\left(1,2\sqrt{2}\right)$
$f(x,y)$	$-\frac{4}{3}$	32	20	16	16

From the table, we see that f has a minimum value of $f\left(\frac{1}{3},0\right) = -\frac{4}{3}$ and a maximum value of $f(-3,0) = 32$.