9.7

1. Let $u = \dfrac{x^n}{n+1}$. Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}}\right| = \lim_{n\to\infty}\left[\left(\dfrac{n+1}{n+2}\right)|x|\right] = |x|$, so the radius of convergence is 1.

The series converges for $-1 < x < 1$. At $x = -1$ the series is $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}$, which converges, and at $x = 1$ it is $\sum_{n=0}^{\infty}\dfrac{1}{n+1}$, which diverges. Thus, the interval of convergence is $[-1, 1)$.

5. Let $u_n = \dfrac{(2x)^n}{n!}$. Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{(2x)^{n+1}}{(n+1)!}\cdot\dfrac{n!}{(2x)^n}\right| = \lim_{n\to\infty}\left(\dfrac{2}{n+1}\right)|x| = 0$ for any real x, so the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

7. Let $u_n = (nx)^n = n^n x^n$. Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n (n+1)\,|x| = \infty$ for $x \neq 0$, so the radius of convergence is 0 and the series converges only at $x = 0$.

9. Let $u_n = \dfrac{x^n}{\ln n}$. Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{x^{n+1}}{\ln(n+1)}\cdot\dfrac{\ln n}{x^n}\right| = \lim_{n\to\infty}\dfrac{\ln n}{\ln(n+1)}\cdot|x| = |x|$, so the radius of convergence is $R = 1$ and the series converges for $-1 < x < 1$. At $x = -1$ the series is $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$, which converges, and at $x = 1$ it is $\sum_{n=2}^{\infty}\dfrac{1}{\ln n}$, which diverges. Thus, the interval of convergence is $[-1, 1)$.

11. Let $u_n = \dfrac{e^n x^n}{n}$. Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{u\to\infty}\left|\dfrac{e^{n+1}x^{n+1}}{n+1}\cdot\dfrac{n}{e^n x^n}\right| = \lim_{n\to\infty}\left(\dfrac{n}{n+1}\right)e\,|x| = e\,|x|$, so the radius of convergence is $\dfrac{1}{e}$ and the series converges for $-\dfrac{1}{e} < x < \dfrac{1}{e}$. At $x = -\dfrac{1}{e}$, the series is $\sum\dfrac{(-1)^n}{n}$, which converges, and at $x = \dfrac{1}{e}$ it is $\sum\dfrac{1}{n}$, which diverges. Thus, the interval of convergence is $\left[-\dfrac{1}{e}, \dfrac{1}{e}\right)$.

12. Let $u_n = \dfrac{(-1)^n n!x^n}{2^n}$. Then $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}(n+1)!x^{n+1}}{2^{n+1}}\cdot\dfrac{2^n}{(-1)^n n!x^n}\right| = \lim_{n\to\infty}\left(\dfrac{n+1}{2}\right)|x| = \infty$ for $x \neq 0$, so $R = 0$ and the series converges only at $x = 0$.

15. Let $u_n = \dfrac{(-1)^{n-1}(x-2)^n}{n\cdot 3^n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(x-2)^{n+1}}{(n+1)\,3^{n+1}}\cdot\frac{n\cdot 3^n}{(-1)^{n-1}(x-2)^n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)|x-2| = \tfrac{1}{3}|x-2| < 1$$

$\Leftrightarrow |x-2| < 3$, so the radius of convergence is 3 and the series converges on $(-1, 5)$. At $x = -1$ the series is $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\dfrac{(-1)^{2n-1}}{n}$, which diverges, and at $x = 5$ it is $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n}$, which converges. Thus, the interval of convergence is $(-1, 5]$.

19. Let $u_n = \dfrac{(-1)^n (x+2)^{2n+1}}{(2n+1)!}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+2)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n (x+2)^{2n+1}}\right| = \lim_{n\to\infty}\frac{|x+2|^2}{(2n+2)(2n+3)} = 0$$ for any real x, so $R=\infty$ and the interval of convergence is $(-\infty,\infty)$.

25. Let $u_n = \dfrac{x^n}{n(\ln n)^2}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)[\ln(n+1)]^2}\cdot\frac{n(\ln n)^2}{x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left[\frac{\ln n}{\ln(n+1)}\right]^2 |x| = |x|,$$ so $R=1$ and the series converges on $(-1,1)$. At $x=-1$ the series is $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$, which converges, and at $x=1$ it is $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$, which also converges (see Exercise 9.3.33). Thus, the interval of convergence is $[-1,1]$.

38. $f(x) = \sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2 3^n}$. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2 3^{n+1}}\cdot\frac{n^2 3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)^2 |x-2| = \frac{1}{3}|x-2| < 1$

$\Rightarrow |x-2| < 3$, so the series converges on $(-1,5)$. At $x=-1$ the series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$,

which converges, and at $x=5$ it is $\sum_{n=1}^{\infty}\frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which also converges. Therefore,

the interval of convergence is $[-1,5]$. $f'(x) = \sum_{n=1}^{\infty}\frac{n(x-2)^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(x-2)^{n-1}}{n3^n}$ and

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^n}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-2)^{n-1}}\right| = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n}{n+1}\right)|x-2| = \frac{1}{3}|x-2| < 1 \Rightarrow |x-2| < 3 \Rightarrow f'$

converges on $(-1,5)$. At $x=-1$ the series is $f'(-1) = \sum_{n=1}^{\infty}\frac{n(-3)^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n}$, which converges, and at $x=5$

it is $f'(5) = \sum_{n=1}^{\infty}\frac{n3^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{1}{3n}$, which diverges. The interval of convergence is thus $[-1,5)$.

39. $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ and

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^n}{n+1} \cdot \frac{n}{x^{n-1}}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)|x| = |x|$, so f' converges on $(-1, 1)$. If $x = -1$ the series is $f'(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which converges, and at $x = 1$ it is $f'(1) = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Thus, f' has interval of convergence $[-1, 1)$. $f''(x) = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n}$ and

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{nx^{n-1}}{n+1} \cdot \frac{n}{(n-1)x^{n-2}}\right| = \lim_{n\to\infty} \left(\frac{n^2}{n^2-1}\right)|x| = |x|$, so f'' converges on $(-1, 1)$. At $x = -1$ we have $f''(-1) = \sum_{n=2}^{\infty} \frac{(n-1)(-1)^{n-2}}{n}$, and since $\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = 1$, $\lim_{n\to\infty} \frac{(n-1)(-1)^{n-2}}{n}$ does not exist, and the Divergence Test shows that $f''(-1)$ is divergent. Similarly, $f''(1) = \sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right)$ is divergent, so the interval of convergence of f'' is $(-1, 1)$. It is easy to see that the interval of convergence of f is $[-1, 1]$.

40. $\sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n^2}$. Since $-\frac{1}{n^2} \le \frac{\sin(n^3 x)}{n^2} \le \frac{1}{n^2}$ and both $\sum_{n=1}^{\infty} \left(-\frac{1}{n^2}\right)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge, the Comparison Test implies that the given series converges for all real values of x. $\sum_{n=1}^{\infty} \frac{d}{dx}\left[\frac{\sin(n^3 x)}{n^2}\right] = \sum_{n=1}^{\infty} n \cos(n^3 x)$. Since $\lim_{n\to\infty} n \cos(n^3 x) \neq 0$, the Divergence Theorem shows that this series is divergent. This does not contradict Theorem 2 because the given series is not a power series.

9.8

3. $f(x)=e^x,\ f'(x)=e^x,\ f''(x)=e^x,\ \ldots,\ f^{(n)}(x)=e^x,\ \ldots$

$f(2)=e^2,\ f'(2)=e^2,\ f''(2)=e^2,\ \ldots,\ f^{(n)}(2)=e^2,\ \ldots$

The required Taylor series is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n=\sum_{n=0}^{\infty}\frac{e^2}{n!}(x-2)^n=e^2\left[1+(x-2)+\frac{1}{2!}(x-2)^2+\cdots+\frac{1}{n!}(x-2)^n+\cdots\right].$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{e^2(x-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^2(x-2)^n}\right|=\lim_{n\to\infty}\left(\frac{1}{n+1}\right)|x-2|=0,\text{ so }R=\infty.$$

5. $f(x)=\sin 2x,\ f'(x)=2\cos 2x,\ f''(x)=(-1)^1 2^2\sin 2x,\ f'''(x)=(-1)^1 2^3\cos 2x,\ f^{(4)}(x)=(-1)^2 2^4\sin 2x,\ldots$

$f(0)=0,\quad f'(0)=2,\quad f''(0)=0,\quad f'''(0)=-2^3,\quad f^{(4)}(0)=0,\quad \ldots$

The required Maclaurin series is $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=2x-\frac{2^3}{3!}x^3+\frac{2^5}{5!}x^5-\cdots+\frac{(-1)^k 2^{2k+1}}{(2k+1)!}x^{2k+1}+\cdots.$

Note that we have changed the index variable to k, where $n=2k+1$.

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|=\lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{2k+3}x^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{(-1)^k(2)^{2k+1}x^{2k+1}}\right|=\lim_{k\to\infty}\frac{4}{(2k+2)(2k+3)}|x|^2=0,\text{ so }R=\infty.$$

9. $f(x)=\ln x,\ f'(x)=\frac{1}{x},\ f''(x)=-\frac{1}{x^2},\ f'''(x)=\frac{2}{x^3},\ \ldots,\ f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n},\ \ldots$

$f(2)=\ln 2,\ f'(2)=\frac{1}{2},\ f''(2)=-\frac{1}{2^2},\ f'''(2)=\frac{2}{2^3},\ \ldots,\ f^{(n)}(2)=\frac{(-1)^{n-1}(n-1)!}{2^n},\ \ldots$

The required Taylor series is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{n!2^n}(x-2)^n=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}(x-2)^n.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n(x-2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(-1)^{n-1}(x-2)^n}\right|=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n}{n+1}\right)|x-2|=\tfrac{1}{2}|x-2|<1\text{ or}$$

$|x-2|<2$, so $R=2$.

11. $$f(x)=\frac{1}{1+x}=\frac{1}{2+(x-1)}=\frac{1}{2\left[1+\left(\frac{x-1}{2}\right)\right]}=\frac{1}{2}\cdot\frac{1}{1-\left[-\left(\frac{x-1}{2}\right)\right]}=\frac{1}{2}\sum_{n=0}^{\infty}\left[-\left(\frac{x-1}{2}\right)\right]^n$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-1)^n$$

The series converges for $\dfrac{|x-1|}{2}<1$ or $|x-1|<2$, so $R=2$.

17. $\displaystyle f(x)=xe^{-x}=x\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!}$ with $R=\infty$.

19. $\displaystyle f(x)=x^2\cos x=x^2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n)!}$ with $R=\infty$.

27. $f(x) = \ln\left(1+x^2\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(x^2\right)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}$ with $R = 1$.

45. $\frac{1}{1+x^3} = \frac{1}{1-\left(-x^3\right)} = \sum_{n=0}^{\infty} \left(-x^3\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$, so $\int \frac{dx}{1+x^3} = \int \sum_{n=0}^{\infty} (-1)^n x^{3n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + C$.

49. $\frac{\ln(1+x)}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$, so $\int \frac{\ln(1+x)}{x}\, dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n-1}}{n}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^2} + C$.