聯微作業解答 (6.5, 6.7)

6.5

• Find the derivative of the function.

34. $g(x) = \cos^{-1}(2x-1)$

Sol. $$g'(x) = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-1}{\sqrt{x-x^2}}$$

43. $g(x) = \tan^{-1}x + x\cot^{-1}x$

Sol. $$g'(x) = \frac{1}{1+x^2} + \cot^{-1}x + \frac{-x}{1+x^2} = \frac{1-x}{1+x^2} + \cot^{-1}x$$

54. $y = \sin^{-1}(\frac{\sin x}{1+\cos x})$

Sol. $$y' = \frac{\frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}}{\sqrt{1-(\frac{\sin x}{1+\cos x})^2}} = \frac{\frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}}{\frac{\sqrt{2\cos x + 2\cos^2 x}}{1+\cos x}}$$
$$= \frac{1}{\sqrt{2\cos x + 2\cos^2 x}}$$

• Find or evaluate the given integral.

77. $\int_0^1 \frac{x^3}{1+x^8}dx$

Sol. Let $u = x^4$, $du = 4x^3dx$, $x=0 \Rightarrow u=0$, $x=1 \Rightarrow u=1$
$$\int_0^1 \frac{x^3}{1+x^8}dx = \frac{1}{4}\int_0^1 \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1}u|_0^1 = \frac{\pi}{16}$$

79. $\int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

Sol. Let $u = \sin^{-1}x$, $du = \frac{1}{\sqrt{1-x^2}}dx$, $x=0 \Rightarrow u=0$, $x=\frac{\sqrt{3}}{2} \Rightarrow u=\frac{\pi}{3}$
$$\int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{3}} u\,du = \frac{1}{2}u^2|_0^{\frac{\pi}{3}} = \frac{\pi^2}{18}$$

6.7

- Evaluate the limit using L'Hôpital's Rule if appropriate.

25. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{e^x - e^{-x} - 2x}$

Sol. $\lim\limits_{x\to 0} \dfrac{\sin x - x}{e^x - e^{-x} - 2x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\cos x - 1}{e^x + e^{-x} - 2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{-\sin x}{e^x - e^{-x}} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{e^x + e^{-x}}$
$= -\dfrac{1}{2}$

31. $\lim\limits_{x\to 0} \dfrac{(\sin x)^2}{1 - \sec x}$

Sol.$\lim\limits_{x\to 0} \dfrac{(\sin x)^2}{1 - \sec x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{2\sin x \cos x}{-\sec x \tan x} = \lim\limits_{x\to 0}(-2\cos^3 x) = -2$

40. $\lim\limits_{x\to \frac{\pi}{2}} [(\pi - 2x)\sec x]$

Sol. $\lim\limits_{x\to \frac{\pi}{2}} [(\pi - 2x)\sec x] = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\pi - 2x}{\cos x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to \frac{\pi}{2}} \dfrac{-2}{-\sin x} = 2$

42. $\lim\limits_{x\to 0^+} x^{\sin x}$

Sol. Let $y = x^{\sin x}$, $\ln y = \sin x \ln x$
$\ln(\lim\limits_{x\to 0^+} y) = \lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \sin x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\csc x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\csc x \cot x}$
$= \lim\limits_{x\to 0^+} \dfrac{-\sin^2 x}{x\cos x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0^+} \dfrac{-2\sin x \cos x}{\cos x - x\sin x} = 0$
Hence $\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} x^{\sin x} = e^0 = 1$

51. $\lim_{x\to\infty}(1+\frac{1}{x})^{x^3}$

Sol. Let $y=(1+\frac{1}{x})^{x^3}$, $\ln y = x^3\ln(1+\frac{1}{x})$

$$\ln(\lim_{x\to\infty} y) = \lim_{x\to\infty}\ln y = \lim_{x\to\infty} x^3\ln(1+\frac{1}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x^3}} \overset{\frac{0}{0}}{=} \lim_{x\to\infty}\frac{\frac{\frac{-1}{x^2}}{1+\frac{1}{x}}}{\frac{-3}{x^4}}$$

$$= \lim_{x\to\infty}\frac{x^2}{3(1+\frac{1}{x})} = \infty$$

Hence $\lim_{x\to\infty} y = \lim_{x\to\infty}(1+\frac{1}{x})^{x^3} = e^{\infty} = \infty$